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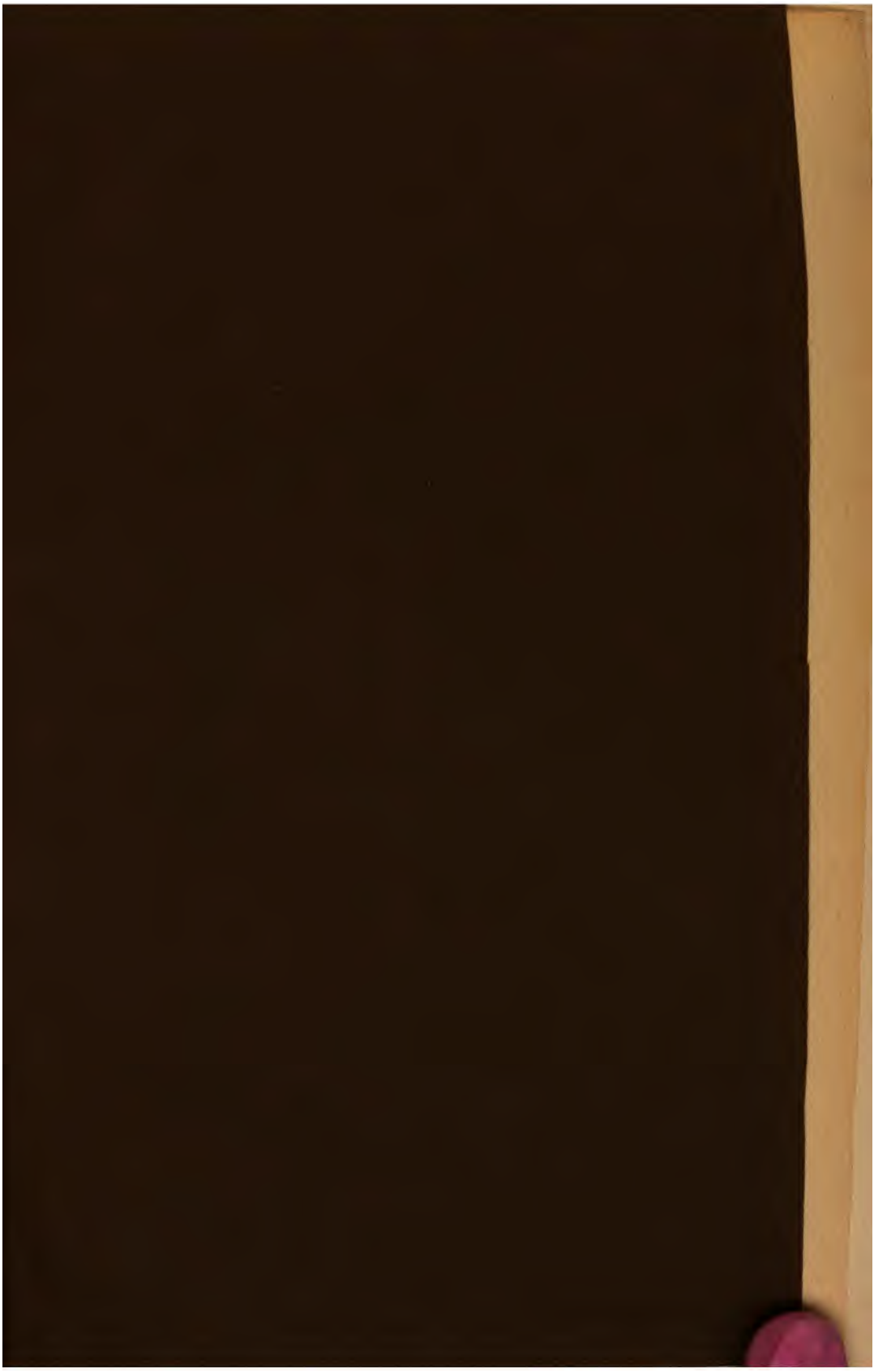
MAXIMUM STRESSES
UNDER
CONCENTRATED LOADS.



MAXIMUM STRESSES
UNDER
CONCENTRATED LOADS.

MAXIMUM STRESSES

UNDER

Concentrated Loads,

TREATED GRAPHICALLY.

— BY —

HENRY T. EDDY, C. E., Ph. D.,

Professor of Mathematics and Civil Engineering in the University of Cincinnati.

ILLUSTRATED BY TWENTY-FIVE FIGURES IN TEXT AND ONE FOLDING PLATE.

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ERRATUM.

On page 39, line 8, for $+ \frac{1}{2} w$, put $-\frac{1}{2} w$.

INTRODUCTION.

1. The object of this paper is to introduce a new graphical method for determining what position a moving train of wheel weights must have in order to produce the greatest stress in any given part of the bridge truss or girder over which the train is passing.

The method which will be here proposed depends principally upon the construction and use of a class of polygons or curves, which we have named reaction polygons. One of these is the line of shears, otherwise called the load line, which has been long used to represent to the eye the distribution of shears or loads in the span; but its graphical possibilities seem to have been little studied. One valuable proposition respecting the load line has, however, been in constant use here since 1875, by Mr. C. L. Strobel, M. Am. Soc. C. E., and his successors in the office of the Cincinnati Southern Railway, who learned it in 1871 as a student of Professor Mohr, then at Stuttgart, now at Dresden. This paper will be mostly devoted to the development of numerous propositions of a character similar to that one.

Reaction polygons, as will appear, are readily constructed graphically, and their properties are such as to give with ease the train positions for maximum stresses as well as to decide which of several maxima is the greatest. The proof of these constructions is given in algebraic form, or, rather, the graphical constructions are merely representations of the algebraic conditions for maxi-

mum stresses. A number of the propositions of this algebraic theory have been long known to the profession,* without mentioning the single case of maximum bending moment before mentioned as having been worked out algebraically and graphically by Professor Mohr before 1871.

The author has enlarged the algebraic theory by applying it to many cases not before treated, especially in cantilevers and double intersection trusses, but, as before stated, the principal object has been to show how the algebraic theory leads to convenient graphical solutions of the equations of condition for maximum stresses.

A general description of this method was given by the author in a paper entitled "A Contribution to the Theory of Maximum Stresses in Bridges Under Concentrated Loads," read before the American Association for the Advancement of Science, at the Buffalo meeting, August, 1886. A paper entitled "Reaction Polygons and their Properties" was also read by him before the same Association at their New York meeting, August, 1887. Both these papers were printed by abstract only.

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* William Cain, "*Van Nostrand's Magazine*," March, 1885; G. F. Swain, *Transactions* Am. Soc. C. E., July, 1887; W. H. Burr, Bridge and Roof Trusses, 3d Ed., 1886; Ward Baldwin, *Engineering News*, September 28th, October 12th, and December 28th, 1889.

PART I.

Maximum Stresses in Simple Girders and Single Intersection Trusses with End Supports, under Concentrated Loads.

2. Let the stepped line $k_1\ h_1\ k_2 \ldots . h_4$ of Fig. 1 represent the relative magnitude and position of a system of wheel-weights, $W_1 \ldots . W_4$, situated at invariable distances from each other. Call this stepped line the *load line* or *weight polygon*.

Suppose this train of wheel-weights to move across a girder of span, $l = A\ B$.

Lay off $k_1\ f_1$, $k_2\ f_2$, $k_3\ f_3$, and $k_4\ f_4$, respectively equal to l, as also $h_1\ g_1$, $h_2\ g_2$, $h_3\ g_3$, and $h_4\ g_4$. Join each of these points f and g to its corresponding k and h respectively, as shown in Fig. 1.

Now, if a girder of span, l, and supported at its ends, occupy the position $A\ B$, then that part of the vertical at A which lies between the lines $f_2\ k_2$ and $f_2\ h_2$ represents so much of W_2 as is supported at A, and that part of the same vertical which lies between $f_3\ k_3$ and $f_3\ h_3$ represents so much of W_3 as is supported at A, and similarly for W_4.

The sum of these quantities is the total load resting upon A, and is equal to the reaction of the pier at A. Let $h_a\ u_a = V'$ denote this reaction at A. Then as the train is moved across the span from right to left (or what is the same thing, as the span is shifted toward the right), let the reaction V' be taken at each point, and let it be laid off upwards from the weight line. The upper extremity u_a of V' will describe some polygon $u_1\ u_2$. The broken line or polygon $u\ u$ described by u_a as the span $A\ B$ is thus shifted, we shall define as the *reaction polygon* of A for the span $A\ B$. This polygon, which is the locus, or path described by u_a is readily constructed as follows: Since the only reaction when A is situated between the verticals at f_1 and f_2 is that due to W_1, it is evident that the vertical distance between $f_1\ k_1$ and $f_1\ h_1$ is the reaction at A; hence $f_1\ f_2'$ is one side of the reaction polygon. The reaction at any point between the verticals at f_2 and f_3 is the sum of that due to the weights W_1 and W_2; hence f_2' and f_3' is a side of the reaction polygon when f_2' be so placed in the vertical through f_3 that the height of f_3', above $f_1\ k_1$, is the sum of the segments of this vertical included within the triangles $f_1\ k_1\ h_1$ and $f_2\ k_2\ h_2$. The polygon extends thence to u_1, so situated in the vertical at W_1 that $h_1\ u_1$ is equal to the sum of the segments of this vertical which are included within the triangles $f_2\ k_2\ h_2$, and $f_3\ k_3\ h_3$, etc., etc. There will in general be an angle of this polygon in each vertical in which there is a weight and in each vertical at a distance l to the left of each weight. At each such vertical it will be necessary to take the sum of the segments of this vertical which lie

within all the triangles of the type $f\,k\,h$ which the vertical cuts, a process of summation not difficult to effect graphically with compasses.

The reaction polygon at B, the other extremity of the span $A\,B$, can be found from the segments of verticals included within the triangles of the type $g\,h\,k$ by a similar process of summation. It is, however, quite unnecessary to actually perform this second summation, for, if we denote the reaction at B by $V'' = h_b\,v_b$, it is evident that $V' + V''$ is the total load upon the span $A\,B$. Hence $u_a\,v_b$ must be horizontal, *i. e.*, v_b is situated on the horizontal through u_a at a distance $l = A\,B$ towards the right. The reaction polygon for the point of support B then differs from that for A only in being displaced a distance l to the right of that for A. As they differ only in horizontal position, they may be said to be parallel. The horizontal line $u_a\,v_b$ may be called the line of *zero shear* for the span $A\,B$. The fitness of this name is evident when we notice that the shear at any assumed point of the span $A\,B$ is measured by the vertical distance at that point from the line of zero shear to the stepped weight line $h\,k$. The maximum shear, positive or negative, at any point C of the span $A\,B$, may be readily found by trial thus:

Cut a rectangular card of width $A\,C$, and call its upper edge $A'\,C'$. Now, keeping $A'C'$ horizontal, slide A' along the reaction polygon $u\,u$. The upper edge $A'C'$ may be regarded as part of a movable line of zero shear. Find by trial the position at which C' is farthest above (or below) the weight line. This will give the position of $A\,C$ for the maximum (or minimum) shear at C. Such trial is simplified by reason of the fact that the maximum shear occurs just in front of one of the forward wheels, near the head of the train, a proposition which will be established later on. It is unnecessary to determine the minimum shear, as it will be surpassed in magnitude by the maximum which occurs near the head of the train in passing over the span in the opposite direction, which is consequently the true minimum. The question of maximum shear will, however, be considered in a more general manner and more at length hereafter.

The maximum value of V' as the train passes over the girder can be readily determined by inspection, as it is the greatest vertical distance from the reaction polygon $u\,u$ to the weight line $h\,k$; this is also the maximum shear at and near the extremity A of the span $A\,B$.

The position of the center of gravity of the total load upon the span $A\,B$ may be readily constructed, as follows:

Draw $h_a\,h_b$ cutting $u_a\,v_b$ the line of no shear at F, then this point F divides the span $u_a\,v_b$ proportionally to the reactions V' and V''. Now lay off $u_a\,G = v_b\,F$, then the point G divides the span inversely as the reactions V' and V'', consequently the center of gravity of the total load upon the span $A\,B$ is in the vertical through G.

Fig. 1

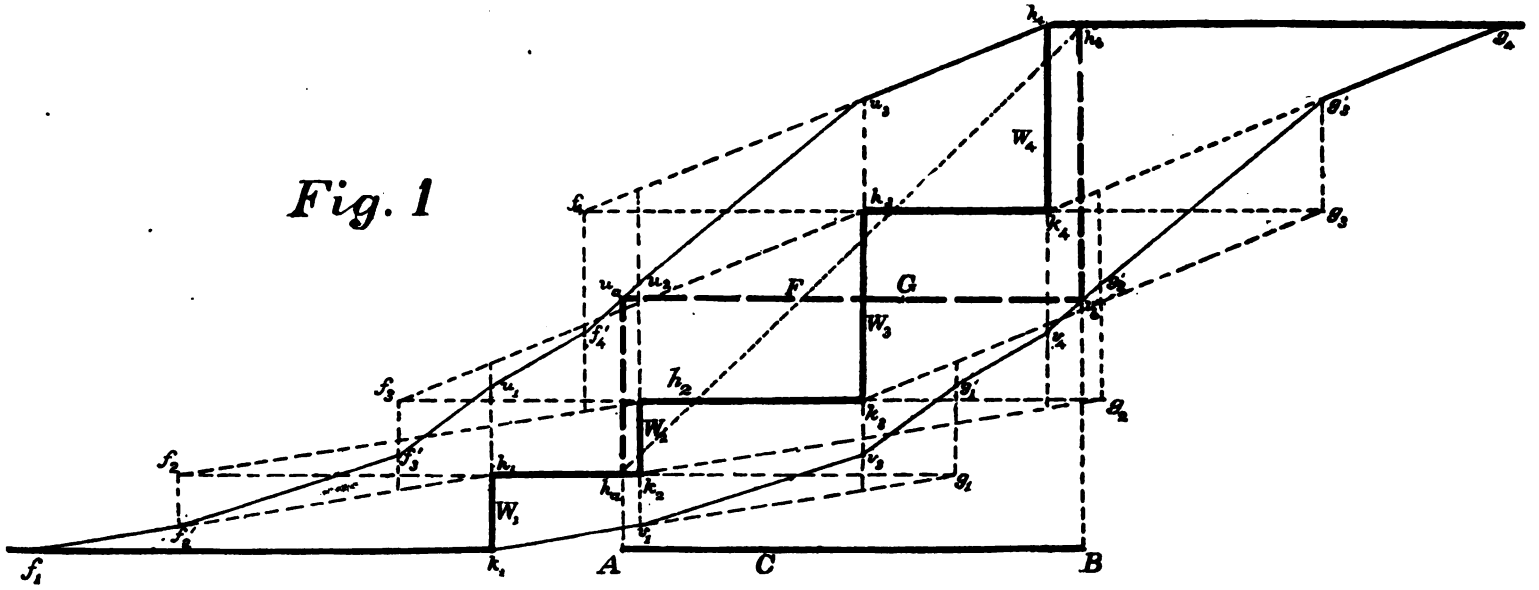
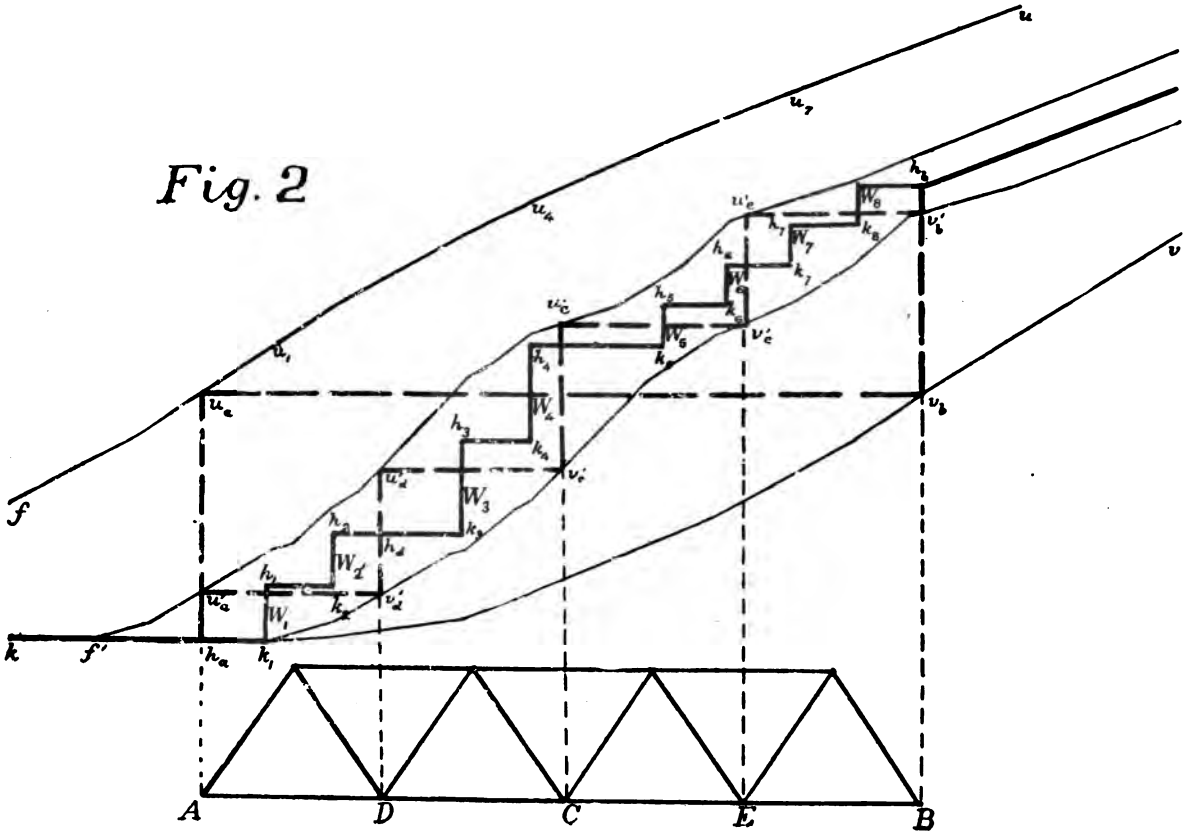


Fig. 2



3. In any single intersection truss the total load carried by the stringers to any pin point, as well as the shear in the stringers of any panel, can be found for any position of a given train by help of reaction polygons, as follows :

Let the stepped weight $h\,k$ of Fig. 2 represent the given train, which may be taken to consist of a locomotive and tender followed by a uniform unlimited train.

Suppose this train to stand upon a single intersection truss of length $l = A\,B$, whose panels are each of length $p = A\,D$. Draw the reaction polygons $u\,u$, $v\,v$, for the span l, and $u'\,u'$, $v'\,v'$, for the span p. Now if the truss be taken in the position $A\,B$, the total reaction at A due to the load upon the span $A\,B$ will be $h_a\,u_a$, and similarly the reaction at A due to the load upon the panel $A\,D$ will be $h_a\,u_a'$.

Likewise, $h_b\,v_b$ is the reaction at B of the total load upon $A\,B$, and $h_b\,v_b'$ that of the load upon the panel $E\,B$. Hence $u_a\,u_a'$ is the shear in the web members of the panel $A\,D$. Again, $h_d\,v_d'$ is the reaction at the pin point D due to the wheels upon $A\,D$, and $h_d\,u_d'$ is that due to the wheels upon $D\,C$.

Hence, $v_d'\,u_d'$ is the load transmitted by the stringers in the panels $A\,D$ and $D\,C$ to the pin at D. Similar statements hold respecting the loads supported at the pin points C and E, viz.: $v_c'\,u_c'$ is the pin load at C, and $u_e'\,v_e'$ that at E. Also the shear in the web members of any panel is the vertical distance between the line of zero shear $u_a\,v_b$ for the span $A\,B$ and the line of zero shear for the panel considered, *i. e.*, $u_a'\,v_d'$ is the line of zero shear in $A\,D$, $u_d'\,v_c'$ that in $D\,C$, etc. The stepped line $u'\,v_d'\,u_d'\,v_c'\ldots$. is the line of zero shears for the panels. Furthermore, it will be easy to find by inspection what position during the passage of the train produces the greatest shear in the end panel $A\,D$, by observing at what point the vertical distance $u_a'\,u_a$ between the reaction polygons $u\,u$ and $u'\,u'$ is greatest. What the position must be to produce the greatest shear in any panel other than the end panel will be considered later. The shear in the stringer of any panel is the vertical distance between the line of zero shear for that panel and the weight line, *e. g.*, in $D\,C$, the shear borne by the stringer is represented by the vertical distance between $u_d'\,v_c'$ and the weight line.

4. The greatest load at a pin point is evidently the greatest vertical distance between the polygons $u'\,u'$ and $v'\,v'$. The position where the pin point experiences its greatest load can easily be found by inspection. In Fig. 2 it is at W_3; hence, during the passage of the train across the truss $A\,B$, whenever the wheel W_3 reaches either of the pin points D, C or E, the load at that pin is then greater than for any other position of the train. But, in order to facilitate the investigation it will be useful to obtain the analytical condition for a maximum pin load as follows: Let the distance from E to any wheel weight be denoted by e and its distance from D by d, both these co-ordinates being reckoned as positive toward the left from the origins E and D.

Then the total pin load R at C may be written—

$$R = \frac{1}{p}\left(\sum_{E}^{C} W e - \sum_{C}^{D} W d\right)$$

in which equation p is the length of either of the equal panels $E\,C$ and $D\,C$, and d is negative for each of the wheels in the panel $D\,C$.

Now, let the train move a small distance $\delta\,x$ toward the left, which movement will change e to $e + \delta\,x$ and d to $d + \delta\,x$, and R to $R + \delta\,R$.

Then we obtain in the new position

$$R + \delta\,R = \frac{1}{p}\left[\sum_{E}^{C} W(e + \delta\,x) - \sum_{C}^{D} W(d + \delta\,x)\right]$$

Hence, by subtraction and division—

$$\frac{\delta\,R}{\delta\,x} = \frac{1}{p}\left(\sum_{C}^{E} W - \sum_{C}^{D} W\right) = \frac{1}{p}(P_e - P_d) \ldots\ldots\ldots\ldots (1)$$

in which equation P_d is the load in the panel $D\,C$, and P_e that in $C\,E$. In order that R may reach a maximum value as the train moves to the left, the quantity $P_e - P_d$ must change from positive to negative. This change can only occur in one way, viz.: Some wheel reaches C which is of such magnitude that in passing from P_e at the right to P_d at the left of C, the last member of equation (1) changes from + to —. The change can occur in this way only, for when a wheel enters $E\,C$ from beyond E, the effect of this is to increase P_e, and when a wheel leaves $D\,C$ at D the effect is to decrease P_d. Either of these occurrences might change the last member of equation (1) from — to +, but not from + to —; and hence, either might cause the value of R to reach a minimum, but not a maximum value.

It has therefore been shown that the only verticals where a maximum value of R can be found are those in which the wheel weights are found, and this restriction will materially facilitate the inspection by which to discover the position of the train when the value of R is a maximum. The maximum value of R might also be shown to be at a wheel when the panels are of unequal length.

5. A system of reaction polygons consists of a number of such polygons belonging to a given train of wheel weights, in which system its several members are constructed to give the reactions for different lengths of span.

The large diagram, Fig. 3, contains several members of such a system due to the train of wheel weights represented by the weight line there shown. The successive reaction polygons are constructed for spans of 5, 10, 15, 20, 25 and 50 feet respectively.

The assumed load consists of two consolidation locomotives and tenders, preceded and followed by a uniform train. The scale of weights is given upon the diagram in pounds, and the scale of distances in feet. It will, however, be convenient to have the scale of distances subdivided more than in Fig. 3 ; it should be divided to single feet or even less than that.

In the solution of the various problems which follow, we shall presuppose that such a system of reaction polygons has been constructed. In practice it will be necessary to draw them upon a sheet permanently mounted, and to a scale sufficiently large to admit of convenient comparisons. For long spans the diagram must include more of the uniform load than is found in Fig. 3. The successive polygons must be constructed for spans which differ so litt'e in length that it shall be possible to interpolate for spans of intermediate length with approximate correctness. It will appear, however, that the successive polygons need not be very near together in order to solve with entire correctness most of the problems proposed. This arises from the circumstance that most of these problems have one peculiarity in common, a peculiarity which was also exemplified in the problem just discussed as to the maximum value of the pin load R. This becomes evident when we notice that the diagram is not employed to determine at this time the actual magnitude of the maximum value of R; its use merely serves to point out which wheel should stand at the pin point under consideration in order that R may reach a maximum at that pin.

The construction is then employed in the first instance in this and other problems in enabling us to select with certainty which one of the wheels produces the maximum. Although in the problem of the maximum load upon a pin it is also feasible to measure with considerable certainty the magnitude of the maximum value of R, in many of the problems to be treated, this will not be possible. Especially in problems pertaining to the maximum value of the bending moments, it will not be possible to learn anything respecting the absolute value of these moments from the reaction polygons, which will be used merely to discover the position of the train producing a maximum. It is evident that no very close approximation will be needed in most cases to distinguish between different wheels and to discover which one fulfills the conditions demanded of a maximum, for the process consists merely in a selection of one from several quantities which differ by finite amounts. For example, it could in general be determined with certainty and ease from Fig. 3 which of the wheels must stand at a pin point to make R a maximum, not only when the panel length is either 5, 10, 15, 20 or 25 feet, but also when it has any intermediate value.

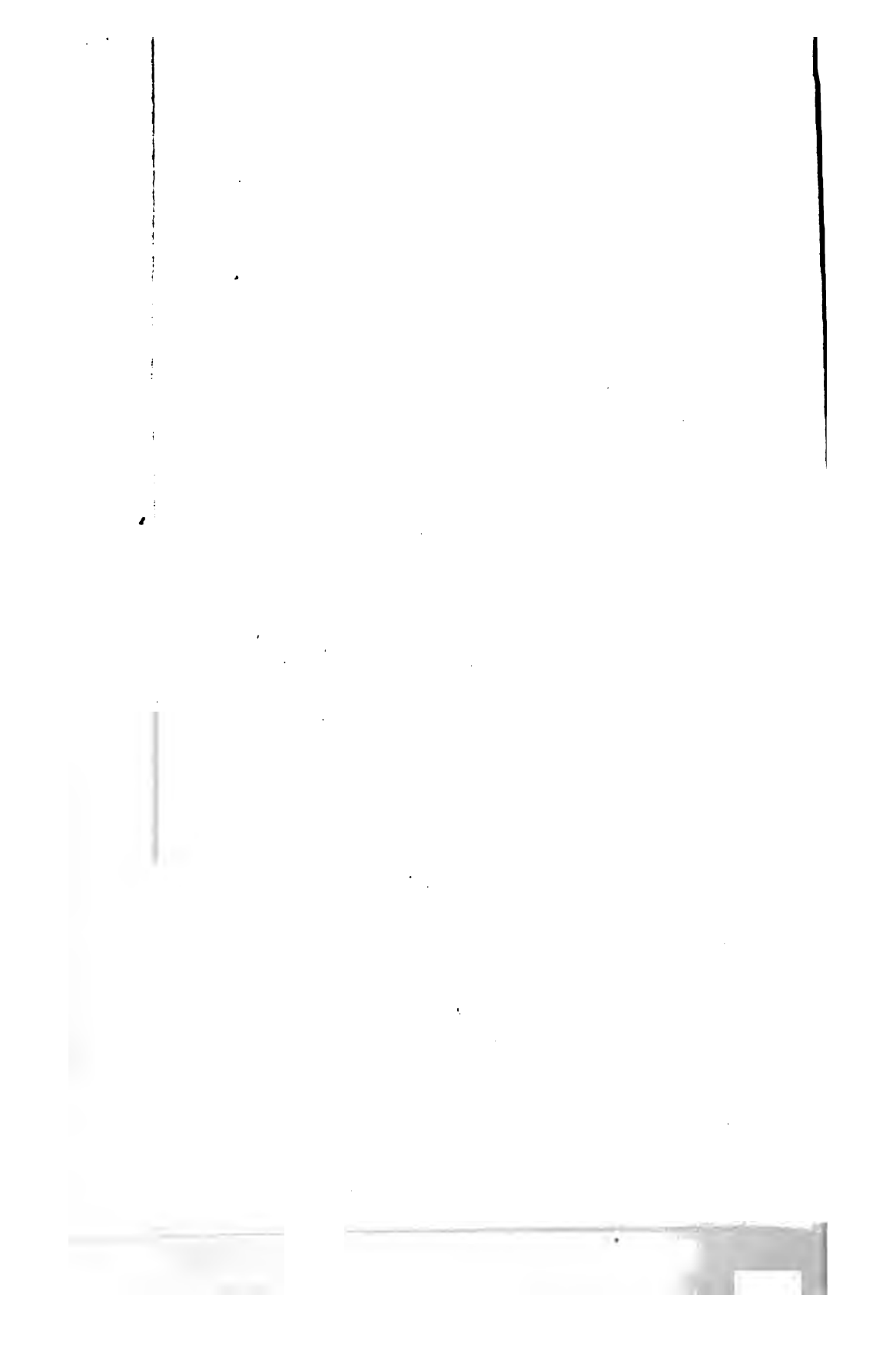
It will be noticed that the reaction polygons approach in character to regular curves as the spans become longer, and that the successive members also become more and more nearly parallel to each other.

From this it appears that the accuracy of interpolation is greater for long spans than for short ones, provided the successive polygons are those for spans which increase successively by equal increments of length. Now, the labor of constructing reaction polygons increases as the span increases. It is advisable, then, in constructing a permanent sheet, similar to Fig. 3, for practical use, to make the polygons near together for short spans and further apart for longer spans.

It may be noticed that the weight line is itself is the reaction polygon for a span whose length is zero, and hence the discussion of the properties of the weight line constitutes an important part of the subject under consideration. In particular, many of the principal problems respecting maximum stresses in the various members of the single intersection truss are solved by the weight line alone. It is further to be noticed that this method will find its field of greatest usefulness where a number of girders or bridges are to be computed for the same specified train of wheel weights, for then a single sheet, containing a system of reaction polygons due to that train, will suffice for any number of determinations. The permanent sheet of reaction polygons will be much more useful if drawn upon cross section or profile paper, as the ruling will very essentially facilitate its use.

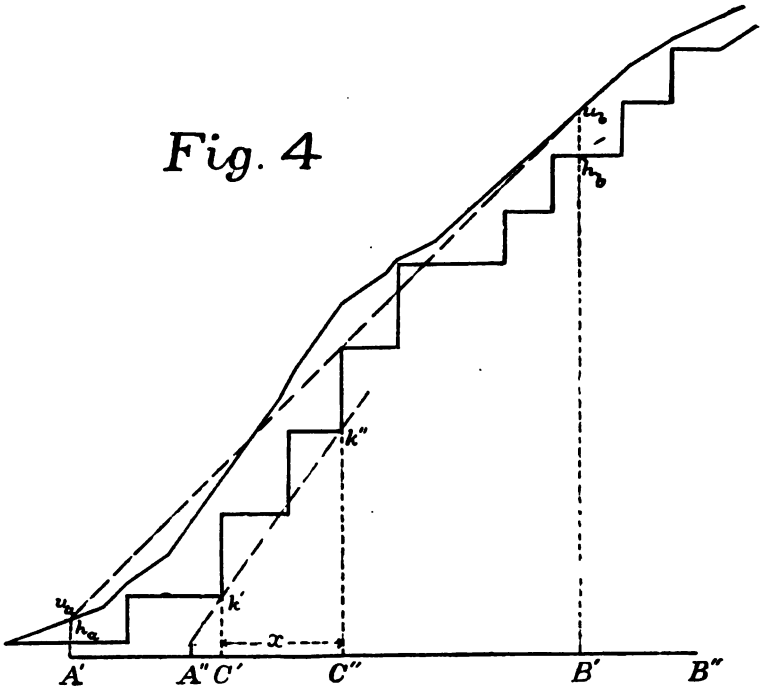
Moreover it will be quite unnecessary to put upon the permanent sheet more than one of the pair of like reaction polygons for each length of span, say, those at the pier A, all of which lie above the weight line. If, for example, it should then be required to find the greatest value of R, it would merely be necessary to use a rectangular card upon which the panel length $A\ D$, Fig. 2, is laid off on the lower edge. Call the lower right-hand corner of the card v_d', and lay off $u_a'\ v_d' = A\ D$. To use the card, slide it on the permanent sheet in such a manner that while the lower edge remains horizontal the point u_a' moves along the reaction polygon, due to the span $A\ D$. The point u_d' will then be found where the right edge cuts $u'\ u'$. Inspection will readily show at which wheel-weight the right-hand edge must be placed in order that the magnitude of $R = v_d'\ u_d'$ may be greatest.

6. The maximum shear at any point, C, of a simple girder, whose span is $A\ B = l$, is known to occur under one of the wheels near the head of the train. To find the greatest shear at any point, it will only be necessary to effect a comparison between the shears under several of the wheels near the head of the train. By help of the reaction polygons the shear under any wheel may, in general, be readily compared with that under any other wheel. To take this more general case first, let the points, $A\ C\ B$, of the girder occupy initially same position, $A'\ C'\ B'$, as represented in Fig. 4, and let them then be shifted toward the right to a second position, $A''\ C''\ B''$, so situated with respect to the train, which is regarded as stationary, that $A'\ A'' = x$.



This is equivalent to moving the train upon the girder a distance x toward the left.

Fig. 4



Let S'' denote shear at C'', and S' that at C'', then:

$$S'' = \frac{1}{l} \sum_{A''}^{B''} W b'' - \sum_{A''}^{C'} W$$

$$S' = \frac{1}{l} \sum_{A'}^{B'} W b' - \sum_{A'}^{C'} W$$

in which equations W denotes any wheel-weight between the limits indicated, b'' and b' the distances from B'' and B', respectively, to any such wheel, taken as positive toward the left.

By subtraction we find:

$$S'' - S' = \sum_{A''}^{B''} W \frac{b''}{l} - \sum_{A}^{B'} W \frac{b'}{l} - \sum_{A''}^{C'} W + \sum_{A'}^{C'} W$$

But, $A'' B'' = A' B' - A' A'' + B' B''$
and, $A'' C' = A' C' - A' A'' + C C'$

Now, change the limits in accordance with these last equations.

$$\therefore \quad S''-S' = \left[\begin{matrix} B' \\ \Sigma \\ A' \end{matrix} - \begin{matrix} A'' \\ \Sigma \\ A' \end{matrix} + \begin{matrix} B'' \\ \Sigma \\ B' \end{matrix}\right] W\frac{b''}{l} - \begin{matrix} B' \\ \Sigma \\ A' \end{matrix} W\frac{b'}{l}$$

$$- \left[\begin{matrix} C' \\ \Sigma \\ A' \end{matrix} - \begin{matrix} A'' \\ \Sigma \\ A' \end{matrix} + \begin{matrix} C'' \\ \Sigma \\ C' \end{matrix}\right] W + \begin{matrix} C' \\ \Sigma \\ A' \end{matrix} W,$$

as may be readily seen by noticing the segments over which the compound summations extend. Now, evidently, $b'' - b' = x$, and if we place $b'' = a'' + l$, then is a'' the distance from A'' to any wheel, taken as positive toward the left. By the help of these relations the foregoing equation may be reduced to the form

$$\frac{S''-S'}{x} = \frac{1}{l}\left(L' - \begin{matrix} A'' \\ \Sigma \\ A' \end{matrix} W\frac{a''}{x} + \begin{matrix} B'' \\ \Sigma \\ B' \end{matrix} W\frac{b''}{x}\right) - \frac{X}{x} \quad \ldots\ldots\ldots (2)$$

in which equation L' is employed for brevity to denote the total initial load, *i. e.*, the sum of the wheel weights between A' and B', while X denotes the sum of the wheel weights between C' and C''.

It will be noticed that, since all the wheels between A' and A'' are at the left of A'', the value of a'' for each of these wheels is essentially positive.

The last three terms of equation (2) take account respectively of the effect of any wheels which during the motion pass off from the girder $A\,B$ at A, of any which enter upon it at B, and of any which pass C, when the initial and final positions differ by any amount x.

If S' be taken to be the shear at C' just at the left of any one wheel, and S'' that at C'' just at the left of any other wheel, then equation (2) will show which wheel produces the greater shear at C.

In Fig. 4 the vertical height of h_b above h_a is L', the load between A' and B',

also $$\begin{matrix} A'' \\ \Sigma \\ A' \end{matrix} W\frac{a''}{x} = \text{the ordinate } h_a\, u_a,$$

and $$\begin{matrix} B'' \\ \Sigma \\ B' \end{matrix} W\frac{b''}{x} = \text{the ordinate } h_b\, u_b,$$

provided u_a and u_b are situated upon the reaction polygon for a span of length x; for these are the expressions for the reactions at A' and B', the left extremities of the spans $A'\,A''$ and $B'\,B''$. Hence

$$\frac{1}{l}\left(L' - \begin{matrix} A'' \\ \Sigma \\ A' \end{matrix} W\frac{a''}{x} + \begin{matrix} B'' \\ \Sigma \\ B' \end{matrix} W\frac{b''}{x}\right)$$

is the tangent of the angle which the sloping line $u_a\ u_b$ makes with the horizontal line $A'\ B'$, and it therefore may be taken as the measure of its inclination or slope.

Again, X is the total load between C' and C'' and $X \div x$ is the tangent of the angle which $k'\ k''$ makes with $A'\ B'$. If the slope of the line $u_a\ u_b$ is steeper than that of $k'\ k''$ then $S'' > S'$, but if $k'\ k''$ is the steeper then $S'' < S'$.

It is to be noticed that any slight vertical errors in the positions of u_a and u_b, such as might arise in obtaining these points by interpolation in the manner previously mentioned, will have little effect in causing error in the slope of $u_a\ u_b$.

This comparison of the relative slopes of $u_a\ u_b$ and $k'\ k''$ will not require the permanent sheet containing the system of reaction polygons to be defaced by drawing these lines upon it, for by using two rulers or a ruler and stretched thread the comparison can be readily effected.

Equation (2) can be still further utilized to determine the segments of the span in which the maximum shear is dominated by the successive wheels near the head of the train, in case the train is taken to consist of one or more locomotives followed, but not preceded by a uniform train. Such a disposition of load is known to cause the severest shear.

It is sufficiently evident that as the train of wheel weights represented in Fig. 5 enters upon the span at B, the shear for same distance near B is greater under W_1 than under any wheel following it.

In order to find the point where the shear just in front of W_2 is first equal to the shear just in front of W_1, let S_2 and S_1 denote the shears just in front of W_2 and W_1 respectively; then in equation (2) we have $S_2 = S_1$ and $X = W_1$, while x is the distance from W_2 to W_1. Equation (2) may then be put in the form

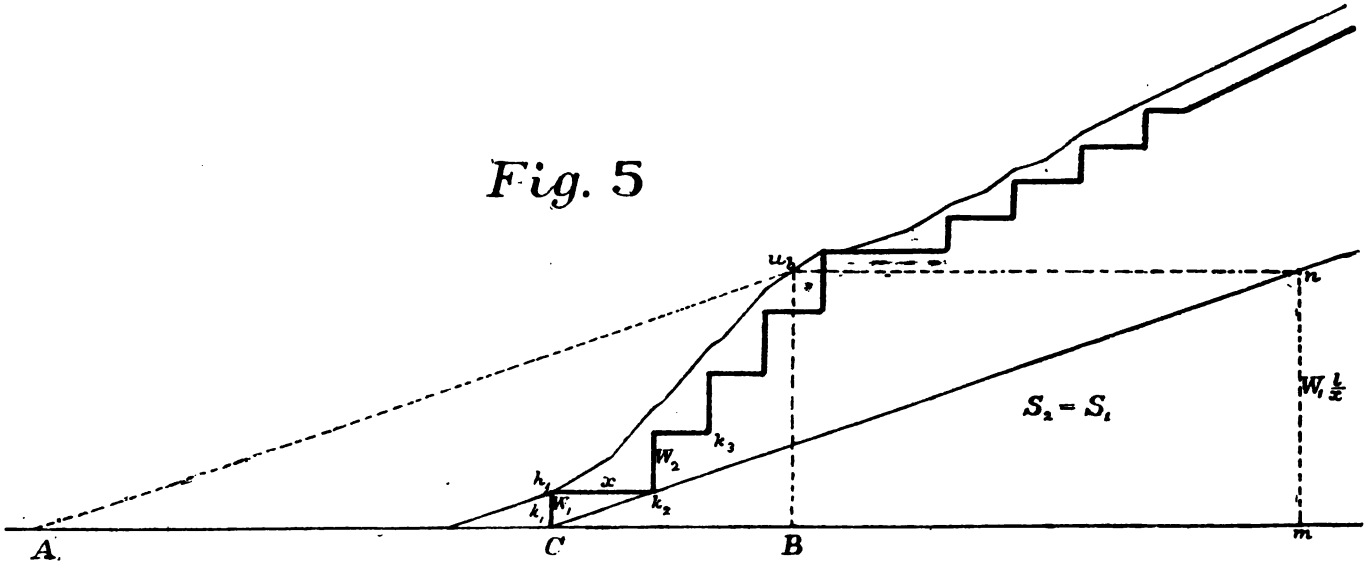
$$L' + \sum_{B'}^{B''} W \frac{b''}{x} = W_1 \frac{l}{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

for the term

$$\sum_{A'}^{A''} W \frac{a''}{l}$$

vanishes because no part of the train is supposed to have reached A'' by the time $S_2 = S_1$. Prolong $k_1\ k_2$, which has the slope $W_1 \div x$ until it cuts the vertical $m\ n$ at a distance from k_1 such that $k_1\ m = l = A\ B$, then is $m\ n = W_1\ l \div x$. Through n draw a horizontal line cutting the reaction polygon for the span x at u_b. Then, if $A\ B = l$, the point k_1 is the point C of the span $A\ B$ where S_2 first becomes equal to S_1, and the maximum shear in the segment $B\ C$ is dominated by the wheel W_1, for the train is then in a position upon $A\ B$, which satisfies equation (2). To the left of C is the segment dominated by W_2.

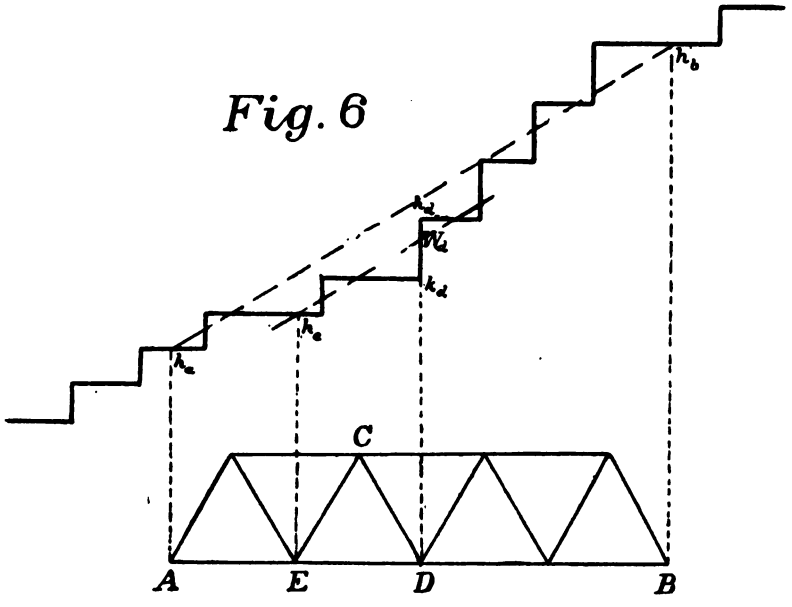
Fig. 5



It should be noticed that if the slope line $S_2 = S_1$ is drawn upon the permanent sheet, then the point B, and so the segment $B\ C$ dominated by W_1, is readily found for any span l without drawing any other lines upon the sheet. Similarly it will be possible to find where S_3, the shear under W_3, first becomes equal to S_2, in case this occurs before W_1 leaves the span; and the distance between the point so found and the point C, found above, will be the segment dominated by W_2.

But there are seldom more than two or three such segments, for the shear is dominated in any ordinary case by one of the wheels very near the head of the train. The greatest shear near A, after one or more wheels has left the span, has been previously discussed.

Fig. 6



It is also to be noticed that when u_b, and so C, is not very accurately determined, this is a matter of small consequence, for the shear under W_1 is very nearly equal to that under W_2 for all points near C, when thus determined. However, it will be well to have the reaction polygon upon the permanent sheet for a span equal to the exact distance between W_1 and W_2.

7. The position of the train for the maximum shear in the web members of any panel $D\ E$ of a single intersection truss $A\ B$ with parallel chords can be found from the weight polygon alone, as may be shown in the following manner:

In Fig. 6 let S denote the shear in the web members of any panel D E, then is S the difference between the shear at C in the span A B, and the shear at C in the span D E, as was shown in connection with Fig. 2. Hence

$$S = \sum_{A}^{B} W \frac{b}{l} - \sum_{A}^{C} W - \sum_{E}^{D} W \frac{d}{p} + \sum_{E}^{C} W,$$

in which equation d is the distance from D to any wheel in the panel D E of length p, and b is the distance from B to any wheel in the span A B of length l.

Now let the train move some small distance δx toward the left, which will change S, b and d respectively into $S + \delta S$, $b + \delta x$ and $d + \delta x$, and we then obtain by subtraction a result which may be written in the form

$$\frac{\delta S}{\delta x} = \frac{L}{l} - \frac{P}{p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

in which equation L denotes the total load upon A B, and P that upon D E. In order that S may reach a maximum value, the last member of equation (4) must change from positive to negative, which can only occur by one of two ways as the train moves toward the left, viz.: either by the entrance of a wheel upon D E at D, or by the departure of a wheel from A B at A; for the former increases P and the latter decreases L. S may reach a maximum value from either of these causes several times during the passage of a train, but is more likely to do so when a wheel enters upon D E at D than when a wheel leaves A B at A, because in equation (4) P has only the small denominator p, whereas L has the denominator l, which is several times larger than p. Hence any change in the value of P exerts a more decisive influence upon the sign of the last member of (4) than the same change in L does.

Whenever a wheel weight W_d arrives at D, as shown in Fig. 6, such that the slope of the line h_e k_d is less, but the slope of the line h_e h_d is greater than the slope of the line h_a h_b, it is evident that the last member of equation (4) will be positive or negative, according as W is regarded as not standing upon the panel D E, or as standing upon it, and the sign of (4) will change from positive to negative, as W_d enters the panel D E.

Whether W_d causes a maximum shear in entering D E, may be discovered most easily by laying down a parallel ruler in the position h_a h_b, and then finding whether the parallel to it through h_e cuts W_d or not. If it does cut W_d, then the last member of (4) changes from + to — as W_d enters upon D E at D, and S is at a maximum for that wheel position.

The work will be facilitated by marking the points A E D B upon the edge of a card and sliding the card upon the permanent sheet to

positions such that D is successively made to lie in the verticals of the successive wheels.

A still more convenient method for discovering the wheel position for a maximum value of S in the end panel has been previously explained in connection with Fig. 2.

Again, when any wheel weight (as $W_a = k_a h_a$ say) leaves the span $A\ B$ at A, there will be two slope lines $h_a\ h_b$ and $k_a\ h_b$ for the span $A\ B$. In case one of these has a greater and the other a less slope than that of the panel slope line $h_e\ h_d$, then S will reach a maximum value by the departure of W_a from the span $A\ B$ at A. Practically, whenever a parallel to $h_e\ h_d$ passed through h_b cuts any wheel weight W_a, then S is a maximum in $D\ E$ with W_a standing at A.

In like manner it may be shown that the shear in $D\ E$ reaches a minimum value either by the departure of some wheel from $D\ E$ at E, or by the entrance of some wheel upon $A\ B$ at B, and more probably in the former case than in the latter.

That part of the graphical process just given which relates to finding the greatest shears in simple trusses was worked out by the author in 1885, and communicated to Ward Baldwin, M. Am. Soc. C. E., who published it in the *Engineering News*, September 28th, 1889, together with some of the other processes used by him in practical work, he giving the author due credit in *Engineering News*, December 28th, 1889, upon learning that the author had not yet published the process.

Now, it is well known that that one of the several maxima values of S which may occur at any point of the span during the passage of the train is in general the greatest which is nearest the head of the train. It will then be most convenient to find those segments of the span in which the maximum value of S is dominated by the successive wheels at the head of the train, which segments may be successively determined as follows: First find where the last member of equation (4) vanishes in case $P = W_1$, *i. e.*, in case $\frac{W_1}{p} l = L$; which last equation serves to determine the value of L in case $P = W_1$, and serves also to determine the position of the train and the length of segment in which the maximum value of S is dominated by W_1. For, in Fig. 7, let $k_1\ i = p$ be the panel length; and let $i\ i_1 = W_1$. Draw the line $k_1\ i_1$ and prolong it to j_1. Lay off $k_1\ m = l$ the length of the span, then by similarity of triangles we have $m\ j_1 = W_1\ \frac{l}{p}$. Draw a horizontal line through j_1: it cuts W_2; and it is evident that so long as W_1, or W_1 and W_2, stand upon the span, but W_3 does not, the last member of equation (4) will be negative or positive, according as W_1 enters the panel under consideration or not, *i. e.*, the maximum value of S is dominated by W_1 in all those panels of the span of which the right-hand panel points are reached by W_1, provided at least one wheel but not more than two

wheels stand upon the span. It is necessary to insert the proviso just made, that at least one wheel shall be upon the span, for otherwise L would vanish and (4) could not change sign. This proviso necessarily excludes from consideration by this method the panel at the right extremity of the span in which S is invariably zero or negative, for when W_1 stands at its right-hand panel point L vanishes. But that is a matter of no consequence, for the greatest shear in the end panel has been otherwise found before.

In Fig. 7, as already shown, the train can only enter upon the span to a distance equal to that between W_1 and W_2 before the maximum value of S will be dominated by W_2. Consequently $A_1 B_1$ represents the extreme position of the span when two wheels and no more have entered as far as possible upon the span. Hence $C B_1$ is the segment dominated by W_1 after leaving out one panel length at the end where, as before mentioned, the shear is either zero or negative. But $C B_1$ is less than a panel length, consequently no panel of this truss is actually dominated by W_1.

Next let $P = W_1 + W_2 = i i_2$, and prolong $k_1 i_2$ to j_2, then by similarity of triangles we have $m j_2 = (W_1 + W_2)\frac{l}{p}$; and as before, when any load not greater than $m j_2$ has entered upon the span, the maximum value of S is dominated by W_2 in all that part of $C B_2$ not already shown to be dominated by W_1. In Fig. 7 this includes all the remaining panels, for they stand at the left of panel points dominated by W_2. This, however, does not prevent the possibility of a shear still greater than this in the last panel at the left, after one or more wheels have left the span, which will be found by the method previously explained in connection with Fig. 2.

None of these constructions require lines to be drawn upon the permanent sheet, as the distances $C B_1$, $C B_2$, etc., can be read off on the cross-section paper when a stretched thread has been put successively in the positions $k_1 j_1$, $k_1 j_2$, etc.

8. The position of a train when it produces a maximum bending moment at any point C of the span $A B$ may also be readily found by help of the weight line alone, as will now be shown.

It is to be noticed that the determination of the maximum bending moment at any point C of the span $A B$ will suffice to determine not only the maximum moment at that point of a simple girder but also the maximum stress in that member of the chord of a single intersection truss which is opposite to any pin point C which carries the live load.

Suppose the live load to consist of one or more locomotives with tenders, followed by an unlimited uniform train, and preceded or not, as the case may be, by a uniform train.

The expression for the value of M, the bending moment at C, may be written thus :

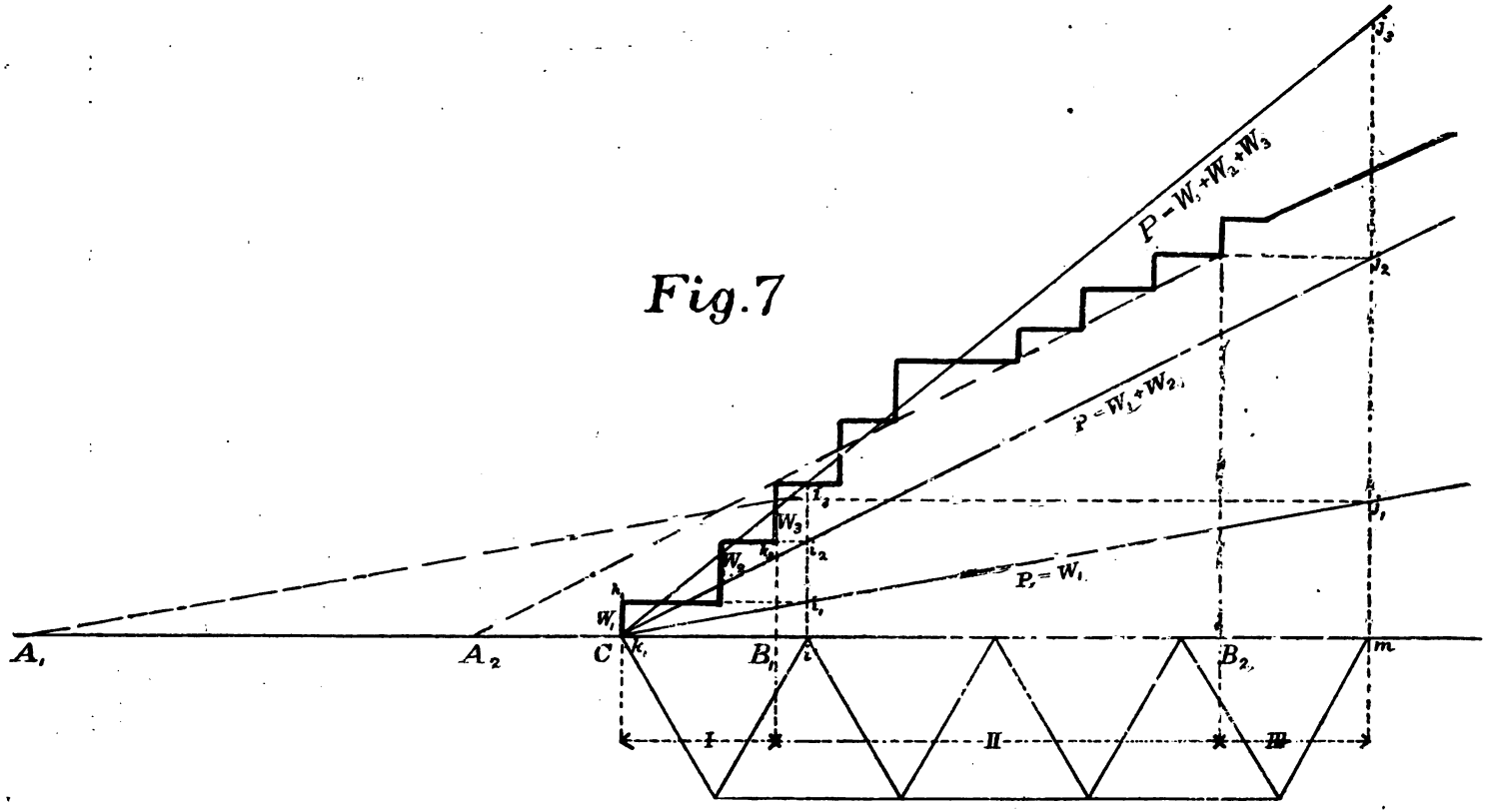


Fig.7

$$M = \frac{c_a}{l} \sum_{A}^{B} W b - \sum_{A}^{C} W c,$$

in which equation c_a is the distance from C to A, b is the distance from B to any wheel weight W upon $A\,B$, and c is the distance from C to any weight W upon $A\,C$.

Now let the train move an indefinitely small distance $\delta\,x$ to the left and we have

$$M + \delta\,M = \frac{c_a}{l} \sum_{A}^{B} W (b + \delta\,x) - \sum_{A}^{C} W (c + \delta\,x),$$

from which equation by subtracting the previous expression we obtain a result which may be written in the form

$$\frac{\delta\,M}{\delta\,x} = L\frac{c_a}{l} - C_a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

an equation in which L denotes the total load upon the span $A\,B = l$, and C_a that upon the segment $A\,C = c_a$.

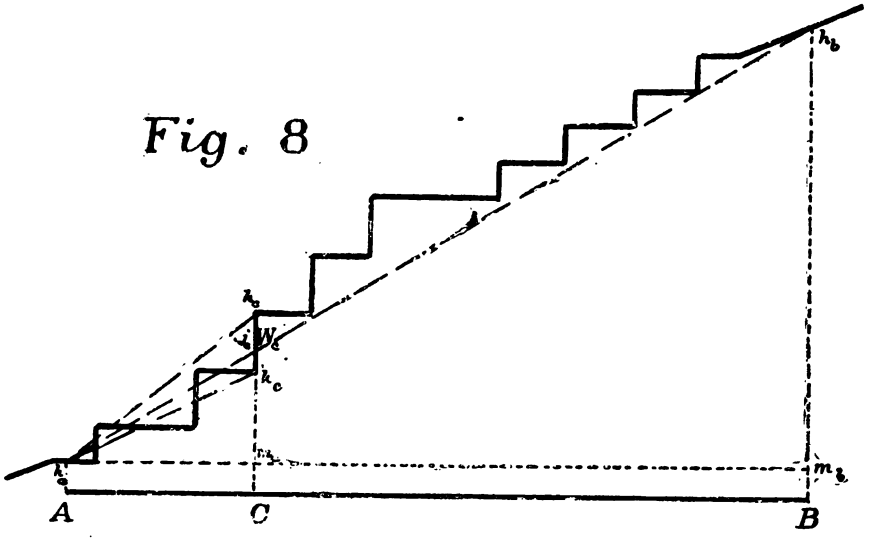
In order that a maximum value of M may occur, the last member of equation (5) must change from positive to negative. Such a change can only occur, during the movement towards the left, by the passage of some wheel weight W_c from the right to the left of the point C, which will increase C_a by an amount W_c.

That the change can occur in this manner is sufficiently evident, but that it can occur in this manner alone may be seen by considering what changes of sign the last member of equation (5) undergoes when wheels either enter upon the span or depart from it. When any wheel enters upon the span at B, its effect is to increase L; but an increase of L cannot cause (5) to become negative. Likewise, when any wheel leaves the span at A, the effect of its departure upon (5) is to diminish both L and C_a by the amount of the wheel weight and thereby on the whole to increase (5), for since $c_a < l$ the departure of the wheel has more influence in causing a numerical diminution of C_a than of $L\frac{c_a}{l}$.

For example, in Fig. 8, for the given position of the span $A\,B = l$, we find the total load $L = m_b\,h_b$ and by similar triangles $L\frac{c_a}{l} = m_c\,j_c$ and C_a is either $m_c\,k_c$ or $m_c\,h_c$ according as W_c is taken to stand just at the right or left of C. Hence the last member of (5) changes from + to − as W_c passes C, and W_c when it reaches C causes a maximum value of M at that point.

In order to find in practice which wheel or wheels produce a maximum moment at C, mark the points $A\,C\,B$ upon the edge of a card, and placing C successively under the wheels upon the permanent sheet, stretch a thread from the point h_a to the point h_b, which points are found directly above A and B, provided $A\,C\,B$ is horizontal. If the

Fig. 8



thread $h_a\ h_b$ crosses the wheel-weight under consideration, then is the moment at C a maximum. If the thread passes entirely above that part of the weight-line in the vertical at C shift the span $A\ C\ B$ toward the right until it cuts the weight at C; if it passes below, shift $A\ C\ B$ toward the left until the thread cuts the weight at C. The method just described for finding the position of the train which produces a maximum moment, was first proposed and taught, so far as is known to the writer, in 1871, by Professor Mohr, then at Stuttgart, now at Dresden, and the useful device of a thread in place of lines is due to Ward Baldwin, C.E. The method has proved to be so useful and expeditious in practice, that the present writer has been induced to make an endeavor to enlarge the circle of our knowledge as to the possible applications of similar graphical constructions, and this present paper is the outcome of his endeavors.

It may frequently be convenient to find the segment of the span in which the maximum bending moment is dominated by a given wheel, say W_c.

This is easily accomplished by supposing C to remain in the vertical at W_c, while the span $A\ B$ is shifted in one direction until $h_a\ h_b$ passes through h_c or k_c, and then in the other direction until it again passes through h_c or k_c. The part of the span $A\ B$ which passes C in shifting $A\ B$ from one of these extreme positions to the other, is the segment dominated by W_c, and may be denoted for convenience by the same number as the wheel.

These segments will overlap each other, *i. e.*, there will often be more than one wheel which causes a maximum moment at a given point.

Which of these maxima is the greater can be discovered by the help of the reaction polygon. as a case of the following more general problem.

9. To determine which of two given positions of the train produces the greater bending moment at any given point C of the span $A\,B$.

Let the points $A\ C\ B$ be taken at first in the position $A'\ C'\ B'$, as shown in Fig. 9; then the bending moment at C' may be written with the same notation, as before in the form.

$$M' = \frac{c_a}{l}\sum_{A'}^{B'} W b' - \sum_{A'}^{C'} W c'$$

in which equation c_a is the distance from C to A, or from C' to A'. Now let the train be supposed to move a distance x to the left, then will the points $A\ C\ B$ be shifted a distance x towards the right upon the diagram to the position $A''\ C''\ B''$ such that $A''\ A' = B''\ B' = C''\ C' = x$.

The moment M'' at C'' may be written:

$$M'' = \frac{c_a}{l}\sum_{A''}^{B''} W b'' - \sum_{A''}^{C''} W c''$$

Then do the following equations hold for each wheel:

$$b'' - b' = x,\ c'' - c' = x.$$

Change the limits in M'' by help of the identities

$A''\ B'' = A'\ B' - A\ A'' + B'\ B''$,

$A''\ C'' = A'\ C' - A'\ A'' + C'\ C''$, and subtract M' from it, etc.

$$\therefore\quad M'' - M' = \frac{c_a}{l}\left[\sum_{A'}^{B'} W\,(b'+x) - \sum_{A'}^{A''} W\,b'' + \sum_{B'}^{B''} W\,b'' - \sum_{A'}^{B'} W\,b'\right]$$

$$-\left[\sum_{A'}^{C'} W\,(c'+x) - \sum_{A'}^{A''} W c'' + \sum_{C'}^{C''} W\,c'' - \sum_{A'}^{C'} W\,c'\right]$$

Cancel terms in b' and c', and in the summations between the limits A' and A'' put

$$b'' = l + a'',\ \text{and}\quad c'' = c_a + a'',$$

in which last two equations a'' denotes the distance of any wheel to the left of A''. Hence, a'' is positive for all wheels between A' and A''.

$$\therefore\quad \frac{M'' - M'}{x} = \left[\sum_{A'}^{B'} W - \sum_{A'}^{A''} W\frac{a''}{x} + \sum_{B'}^{B''} W\frac{b''}{x}\right]\frac{c_a}{l}$$

$$-\left[\sum_{A'}^{C'} W - \sum_{A'}^{A''} W\frac{a''}{x} + \sum_{C'}^{C''} W\frac{c''}{x}\right]$$

Or $$\frac{M'' - M'}{x} = \left(L' - U_{a'} + U_{b'}\right)\frac{c_a}{l} - \left(C_{a'} - U_{a'} + U_{c'}\right) \ldots\ (6)$$

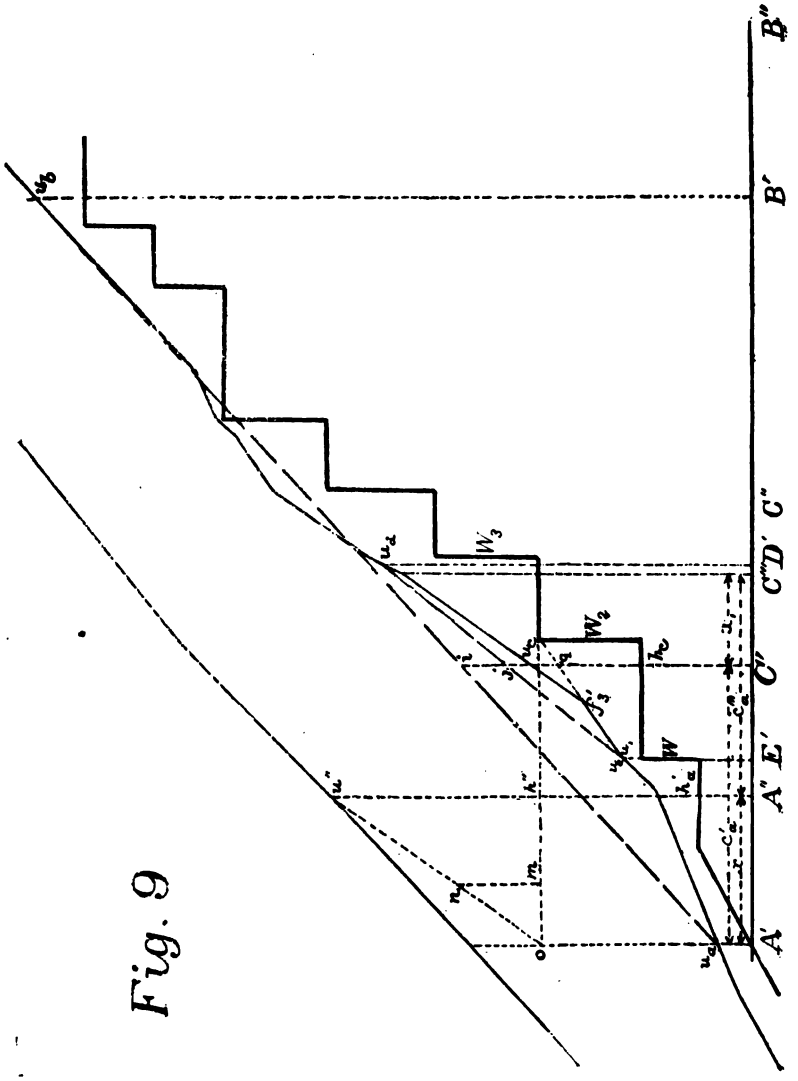


Fig. 9

in which equation the quantity

$$U_a' = \sum_{A'}^{A''} W \frac{a''}{x}$$

is the reaction at the extremity A' of the wheel weights upon the span $A'A''$ of length x. Similarly U_b' and U_c' are the reactions at B' and C' respectively of the load in the spans $B'B''$ and $C'C''$, each of length x.

Hence if $u_a\ u_c\ u_b$ be the reaction polygon for the span x, the altitude of u_a above u_b will represent the magnitude of the first parenthesis in equation (6), and by similarity of triangles the altitude of i above u_a will be that part of the second member of (6) preceding the second parenthesis.

It likewise appears that the second parenthesis is the altitude of u_c above u_a. Hence, finally, equation (6) may be written

$$\frac{M'' - M'}{x} = u_c\ i,$$

from which we see that when u_c lies above $u_a\ u_b$ then $M'' < M'$, but when u_c lies below $u_a\ u_b$ then $M'' > M'$. This will enable us to discover which of two maxima values of M is the greater.

It will be further necessary to investigate the position of the train which will produce a maximum stress in the bars of that chord which carries the live load. To this we now proceed.

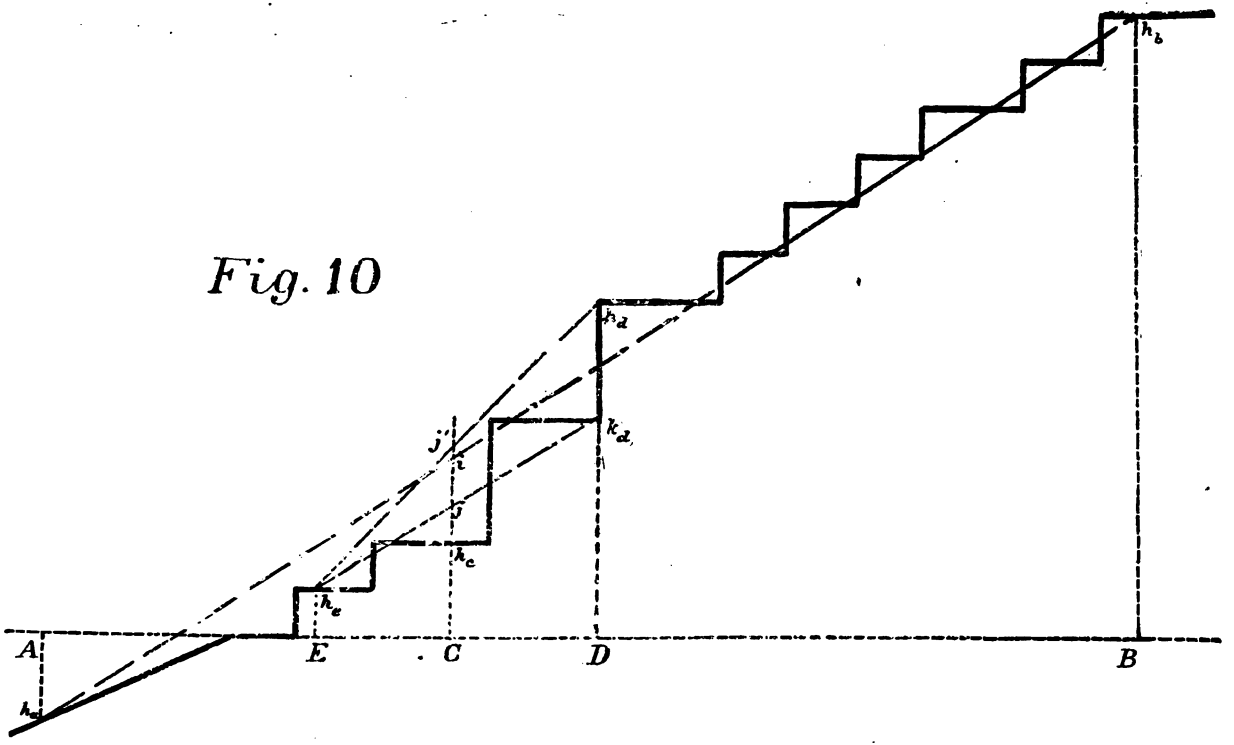
10. The position of the train for a maximum stress in any chord member $D\ E$ of a single intersection truss $A\ B$, when D and E are pin points of the chord which carries the live load, may be readily found by help of the load line alone, thus:

In Fig. 10 let $D\ E$ represent one panel of length p of the single intersection truss $A\ B$, whose span is l, and let the pin point opposite $D\ E$ be in the vertical at C. Denote the distance $C\ A$ by c_a and $C\ E$ by c_e. Also let the distance from B to any wheel in $A\ B$ be b and from D to any wheel in $D\ E$ be d. Then will the bending moment M, which determines the stress in the bar $D\ E$, be the difference between the total bending moment at C, due to the all wheel weights in the span $A\ B$, and the bending moment at C, due to the wheel weights in the panel $D\ E$, for the bending moment due to the latter is borne by the stringers.

$$\therefore \quad M = \frac{c_a}{l} \sum_{A}^{B} Wb - \sum_{A}^{C} Wc - \left[\frac{c_e}{p} \sum_{E}^{D} Wd - \sum_{E}^{C} Wc \right]$$

Let the train move an indefinitely small distance δx toward the left

Fig. 10



and we obtain in the same manner as before

$$\therefore \quad \frac{\delta M}{\delta x} = \frac{c_a}{l} \sum_{A}^{B} W - \sum_{A}^{C} W - \left[\frac{c_e}{p} \sum_{E}^{D} W - \sum_{E}^{C} W \right] \ldots . \quad (7)$$

The last member of (7) must change from positive to negative in order that M may reach a maximum value.

It is clear from the discussion previously had respecting equation (5) that such change cannot occur by reason of wheels entering upon or departing from the span.

Furthermore, it will be found by considering equation (7) that no finite change occurs when a wheel passes C.

Again, whenever a wheel leaves the panel at E, both of the summations in the parenthesis of (7) will be decreased by that wheel weight, but that will have a greater effect upon the last term of the parenthesis than upon the first term, because $c_e < p$, and hence have an effect upon the whole to diminish the second member of (7) and render it negative. Hence a maximum stress in $D\,E$ may be due to the fact that some wheel leaves the panel at E. Also, whenever a wheel enters upon the panel $D\,E$ at D, that may change the sign of the last member of (7) from positive to negative, for it will have no effect except to increase the first term of the parenthesis. Hence a maximum stress in $D\,E$ may exist when some one of the wheels stands at D, or, as has just been shown, at E.

Now, in Fig. 10 let D lie in the vertical of some wheel Wa, and let h_a, h_e, h_c, h_b be the panels of the weight line in the verticals at A, E, C, B, respectively, and let the vertical at C intersect the slope lines $h_a\ h_b$, $h_e\ h_d$ and $h_e\ k_d$ at i, j' and j respectively, then the first two terms of the last member of (7), as has been before shown, are equal to $h\ i$, and the parenthesis of (7) is equal to $h_c\,j'$ or to $h_c\,j$, according as Wd stands upon $D\ E$, or does not stand upon $D\,E$. Hence the last member of (7) changes from + to — when Wd enters the panel at D in case i lies between j' and j. And in general when any wheel stands at D or E such that the point i at the point of intersection of the slope line $h_a\ h_b$ for the span and the vertical at C lies between the two slope lines for the panel, then is the train in a position to produce a maximum stress in $D\,E$.

It will be found quite unnecessary in practice to draw any of the span or panel slope lines upon the permanent sheet.

11. There may evidently be several positions of the train which command a maximum stress in $D\,E$ in the manner just shown. In order to find which one produces the greatest stress in $D\,E$, we shall discuss the more general problem, to discover which of two positions, taken arbitrarily, produces the greater stress in $D\,E$.

Let the quantities M, b, c and d, as used in last discussion, be denoted by M', b', c' and d' in the left of the two positions, and by M'', etc., in the right position, and let x be the distance between the two positions,

just as before in Fig. 9. Now, changing limits, performing subtraction and reductions like those employed in obtaining equation (6), we obtain from the first equation of the last article an equation in which appear not only all the terms which are found in equation (6) having reference to the points $A\ C\ B$, but also an equal number of similar terms with opposite signs having reference to the point $E\ C\ D$. The result may be written in the form

$$\frac{M''-M'}{x} = (L'-U_a' + U_b')\frac{c_a}{l} - (C_a'-U_a' + U_c') - \left[(P'-U_e' + U_d')\frac{c_e}{p} - (C_e'-U_e' + U_c')\right] \ldots\ldots\ldots (8)$$

in which equation P' is the total load upon the panel in the position $D'\ E'$, and C_e' is that upon $C'\ E'$, while U_c', U_d', U_e' are the reactions at the left extremities respectively of the spans $C'\ C''$, $D'\ D''$, $E'E''$, each of length x.

The first half of the last member of (8), referring to the points $A'\ C'\ B'$ is expressed, as has been already shown by the vertical distance $u_c\ i$ in Fig. 9; while the last half of it, referring to the points $E'\ C'\ D'$, as expressed by the vertical distance $u_c\ j$, provided u_a, u_e, u_d, u_b is the reaction polygon of the span x. Hence it appears that when j lies above i, $M'' < M'$, but when j lies below i, $M'' > M'$. The application of this method will enable us to discover which of the two positions, each of which produces a maximum stress in $D\ E$, produces the greater maximum.

We here omit, as of comparatively little practical interest, a comparison of the stresses in any two bars of the chord which carries the live load in a single intersection truss under any assumed positions of the load, such, for example, as produce maximum stresses in those bars. The method bears the same relation to that developed from equation (8), as that of (9) does to (6).

12. This part of the discussion will be concluded by the consideration of a somewhat more general problem: to discover whether the bending moment at one point C' of the span in one position $A'\ B'$ is greater or less than the bending moment at any other point C''' of the span, when the span is shifted to any other position $A''B''$.

Let x_1, Fig 9, denote $C'''C'$ the distance along the train between the points at which the moments are taken, and as before let x denote the distance $A''A' = B''B'$ through which the span is shifted; then is $x - x_1 = C''C'''$ the distance along the girder between the points where the moments M''' and M' are taken. Also let $C'\ A' = c_{a'}$ and $C'''A'' = c_{a'''}$. Then, as before, the moment at C' may be written:

$$M' = \frac{c_{a'}}{l}\sum_{A'}^{B'} W\,b' - \sum_{A'}^{C'} Wc';$$

and that at C''':

$$M''' = \frac{c_a'''}{l} \sum_{A''}^{B''} W b'' - \sum_{A''}^{C'''} W c'''.$$

Also $c_a' + x_1 = c_a''' + x,$

or $c_a''' = c_a' - (x - x_1),$

and $b'' - b' = x = a'' - a',\ b'' = a'' + l,$

$c''' = c_a''' + a'' = c' + x_1 - x + a'',$

$c''' = c + x_1 = c'' + x_1 - x,$

$$\therefore M''' - M' = \frac{c_a'}{l}\left[\sum_{A'}^{B'} W b'' - \sum_{A'}^{A''} W b'' + \sum_{B'}^{B''} W b''\right]$$

$$- \frac{x - x_1}{l} \sum_{A''}^{B''} W b'' - \frac{c_a'}{l} \sum_{A'}^{B'} W (b'' - x)$$

$$- \left[\sum_{A'}^{C'} W c''' - \sum_{A'}^{A''} W c''' + \sum_{C'}^{C'''} W c''' - \sum_{A'}^{C'} W (c''' - x_1)\right].$$

Cancel two pairs of terms, substitute values of c''', etc., rearrange, and we have

$$M''' - M' = \frac{c_a'}{l}\left[x \sum_{A'}^{B'} W - \sum_{A'}^{A''} W (l + a'') + \sum_{B'}^{B''} W b''\right]$$

$$- \frac{x - x_1}{l} \sum_{A''}^{B''} W b'' + \sum_{A'}^{A''} W (c_a' + a'' + x_1 - x)$$

$$- \sum_{C'}^{C'''} W (c'' - x + x_1) - (x - x + x_1) \sum_{A'}^{C'} W,$$

from which equation we finally obtain the result:

$$\frac{M''' - M'}{x} = \frac{c_a'}{l}\left[\sum_{A'}^{B'} W - \sum_{A'}^{A''} W \frac{a''}{x} + \sum_{B'}^{B''} W \frac{b''}{x}\right]$$

$$- \left[\sum_{A'}^{C'} W - \sum_{A'}^{A''} W \frac{a''}{x} + \sum_{C'}^{C'''} W \frac{c''}{x}\right] + \sum_{C'}^{C'''} W \frac{c''}{x}$$

$$- \frac{x - x_1}{x}\left[\sum_{A''}^{B''} W \frac{b''}{l} - \sum_{A''}^{C'''} W\right] \ldots\ldots\ldots\ldots (9)$$

As previously explained, the first two group terms in the last member of (9) are together expressed by $u_c\ i$ of Fig. 9 in case $u_a\ u_c\ u_b$ is the reaction polygon for the span whose length is x. The last paren-

thesis, in equation (9), is the shear at C''' in the span $A''\ B''$, in which the first summation is the reaction at A'' of the total load upon $A''\ B''$, and is expressed by $h_a''\ u''$ in case u'' is upon the reaction polygon for a span of length $l = A''\ B''$, while the last summation is the total load resting on $A''\ C'' = h_a''\ h''$, hence the magnitude of the quantity within the last parenthesis is expressed by $h''\ u''$. Now lay off $h''\ o = x$, and $h''\ m = x_1$, then by similar triangles it appears that the value of the last group term in (9) is $m\ n = \frac{x - x_1}{x_1}(h''\ u'')$. The only term not yet interpreted is the one without the parenthesis. It signifies the reaction at C' of all the wheels resting on the span $C'\ C''$, which are situated between C''' and C', and its value is the segment of ordinate at C', included between u_c and $u_1\ f_3'$, that is,

$$\sum_{C''}^{C'''} W\frac{c''}{x} = q\ u_c$$

Hence equation (9), of which equation (6) is a particular case, may be interpreted in a manner analogous to (6), thus:

$$\frac{M''' - M'}{x} = q\ i - m\ n$$

When i falls below q then is $q\ i$ negative; and when $x_1 > x$ then will m be at the left of o, n will fall below $m\ n$ and $m\ n$ will be negative.

In the case shown in Fig. 9 it appears that M''' the moment at C''' in the span when situated at $A''\ B''$ is greater than M' at C' in $A'\ B'$.

The foregoing demonstration and construction might at first sight appear to be of purely theoretical interest, but it finds important application in determining how many wheels rest on a stringer when it is subjected to the greatest bending moment possible during the passage of the train. For example, if a stringer be of such length as to take only the three of the heaviest wheels at once, it may nevertheless suffer a greater bending moment when only two of them are upon it, by reason of their being nearer the center. Then is x_1 the distance apart of the two wheels under which the maximum moments M''' and M' occur respectively. It will frequently occur that $x_1 = 0$. The point C''' in the span $A''\ B''$ and C' in $A'\ B'$ where each given number of wheels produces its greatest bending moment can easily be found, as will be seen from the investigation which follows respecting limited sets of wheels, all of which remain upon the span.

In applying the general method of graphically comparing any two bending moments with each other to the case just mentioned (when a wheel stands at the point C'''), it might not at first sight be clear whether this wheel should be included in the summation from A'' to C''' or in that from C'''' to C''. But an examination of the question

will show that it makes no difference which summation it is included in, for a variation of x_1 such as so cause C''' to pass a wheel weight and so pass it from A'' C''' to C'' C'' makes no sudden change in the value of (9); for when a wheel is at C''' we have for that wheel $c''=x-x_1$. The value of (9) therefore varies continuously with x or x_1.

13. A few problems will now be discussed which relate to the bending moments in a simple girder carrying a given limited set of wheels at invariable distances from each other, all of which remain upon the girder. These problems will serve especially to display the advantage of using the weight line and certain of its modifications, as applied to the case of a limited set of wheels. The case is necessarily less complicated than that of an unlimited train, treated heretofore, and hence it admits, in some particulars, of a more complete solution than does the more general case.

The first of these problems is to subdivide the span A B into segments I, II, III, etc., in which the maximum bending moment is dominated by the successive wheels W_1, W_2, W_3, respectively, which constitute the set.

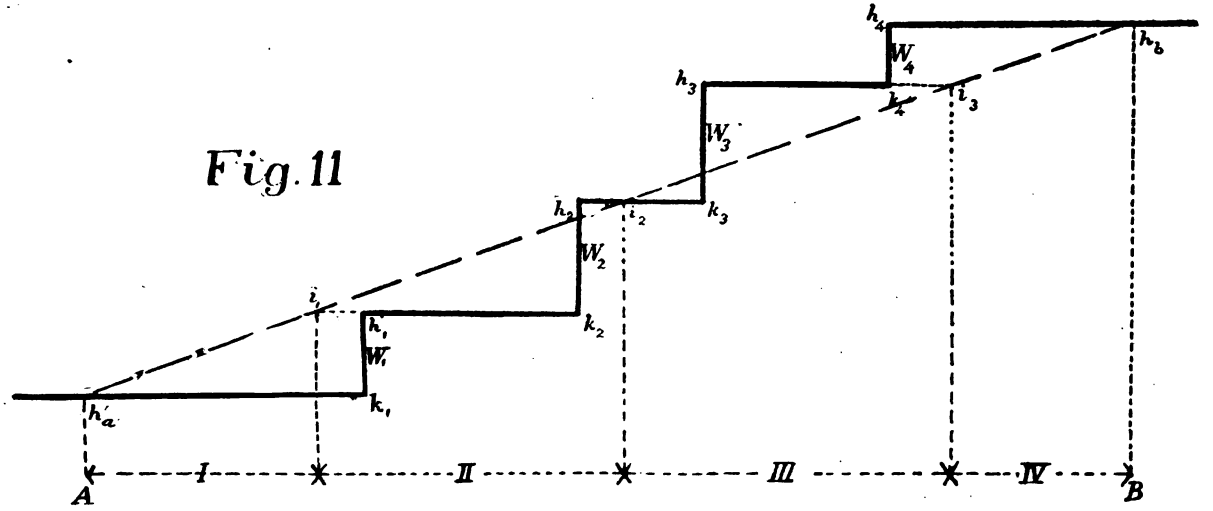
Let the wheel weights W_1 to W_4 represented in Fig. 11 all remain upon the span A B; and in any position, assumed at will, draw the slope line h_a h_b. Call the points where it intersects h_1 k_2, h_2 k_3 and h_3 k_4, i_1, i_2 and i_3, respectively.

If the set be moved to such a position that W_1 stands at i_1 it is evident that the last member of equation (5) would then vanish in case C were taken at i_1, and in case C were taken at any point of the span at the left of i_1, the last member of (5) would change from positive to negative as W_1 moves from right to left of it. Moreover, if W_2 and C be moved to i_2 the last member of (5) would again vanish, and for any point between i_2 and i_1 the last member of (5) would change from positive to negative as W_2 moves from right to left of it. Hence in segment I, extending from A to the vertical at i_1, the maximum moment is dominated W_1. Similarly segment II extends from the vertical of i_1 to that of i_2, etc.

From similarity of triangles it is evident that the lengths of the segments I, II, III, IV, are proportional to the wheel weights W_1, W_2, W_3, W_4; and in general the lengths of the segments of the span in which the moments are dominated by the successive wheels of the set are proportional to the weights of those wheels, provided only that it is physically possible for each wheel to traverse the entire segment which it dominates without thereby causing any other wheel of the set to move off from the span.

If, for example, segment I is shorter than the distance between W_1 and W_2, then W_2 could not move as far as i_1 before W_1 would leave the span: but W_2 would cause the maximum moment in all that part of segment II which it can traverse while W_1 is upon the span. It would

Fig. 11



still be an open question as to the remainder of segment II whether the greater moment would be caused at any point of it by W_1 or by W_2 with W_1 off the span, or possibly by some other wheel as it traverses it, but it is a question which could be settled by Art. 9. Similar statements hold respecting any other segment when the entire segment cannot be traversed by its dominating wheel without causing some other wheel to leave the span.

14. To discover the point of the span where each wheel of a given set, all of which remain upon the span, causes a greater bending moment than it does at any other point. The bending moment M at any point C, as seen previously, is expressed by the equation

$$M = \frac{c_a}{l} \sum_{A}^{B} W b - \sum_{A}^{C} W c.$$

Let the point C be taken at that wheel, W_c say, whose wheel position of greatest moment we wish to find; then is c_a the distance from W_c to A. Let the set of wheels move an indefinitely small distance δx toward the left, then

$$M + \delta M = \frac{(c_a - \delta x)}{l} \sum_{A}^{B} W (b + \delta x) - \sum_{A}^{C} W c.$$

Subtract the former of these equations from the latter, and neglect the small term of the second order containing δx^2. The result may be put in the form

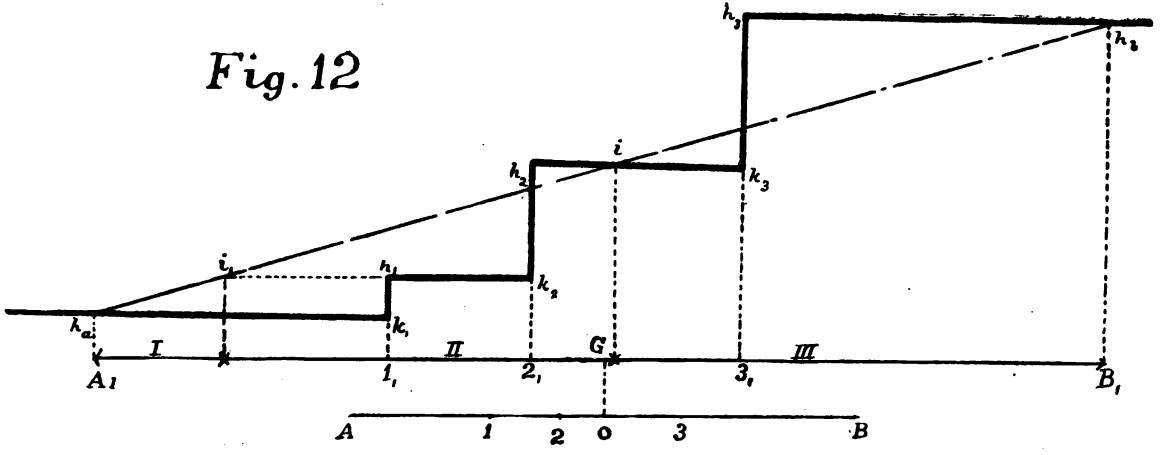
$$\frac{\delta M}{\delta x} = \frac{L}{l} \left(c_a - b_g \right) \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

in which equation L denotes the sum total of the wheel weights upon the span, and b_g is the distance of G, their center of gravity, from B. Hence M reaches a maximum when $c_a = b_g$, which shows that the point where any wheel W_c produces its maximum moment is situated as far from A as the center of gravity of the load is from B, or to state it more simply, the wheel W_c and the center of gravity of the load L must be equidistant from the middle of the span in opposite directions.

These two propositions respecting the maximum bending moments due to a limited set of wheels are, it is believed, original with Culmann, who published them in his *Graphische Statik*, First Edition (1866), pages 137 and 138; Second Edition (1875), page 351, but the method of development and presentation was unlike that which has been here employed.

The wheel position of maximum moment for the span $A\,B$ may be constructed very simply by help of the weight line as follows: Let the set of wheels which remain upon the span $A\,B$ be represented by the weight line $k_1 \ldots . h_3$, as shown in Fig. 12, and let G be the position of their center of gravity.

Fig. 12



It has been shown previously how to find the position of G by using a reaction polygon for any length of span in which these wheels and no others stand. But in case no such reaction polygon has been already constructed, it is to be noticed that the construction for finding the position of the center of gravity of these wheels will be most simple if a span is assumed for that purpose of the same length as the wheel base, so that one extremity is under the first wheel and the other under the last wheel of the set.

Now, having found the position of the vertical at the center of gravity G, it would only be necessary, as has been just shown, to move the train until G is as far from O the middle of the span as W_1 is from O, in order that W_1 should stand in its position of maximum moment. It is evident that W_1 must then stand at 1, a point such that the distance $O1$ is one-half of $G1_1$. Similarly, the point 2 where W_2 produces its maximum moment is situated at a distance $O2$ from O, equal to one-half of $G2_1$, and so on from the other wheels. It will, however, be more simple to lay off $A_1\ G$ and $B_1\ G$ each equal to $A\ B$. Then if $A_1\ B_1$ be taken instead of $A\ B$ to represent the span (on twice the scale of $A\ B$), the point G will be the middle of the span $A_1\ B_1$, and the wheel positions $1_1\ 2_1\ 3_1$ will then be the points where the wheels W_1, W_2, W_3, respectively produce their maximum moments, provided the wheels can reach those points without causing any wheel of the set to leave the span.

15. To discover which wheel of the set produces a greater bending moment in its position of maximum moment than either other wheel produces in its position of maximum moment.

Suppose the span to be subdivided into the segments I, II, III, etc., in which the maximum moment is dominated by the wheels W_1, W_2, W_3, etc., respectively. These segments will be proportional to the weights of the respective wheels, as was shown in Art. 13.

Now, in Fig. 12 it is seen that 1_1 the point where W_1 produces its greatest moment in the span $A_1\ B_1$ lies in segment II, consequently W_2 will cause a greater moment at 1_1 than W_1 does at that point; but W_2 produces a greater moment at 2_1 than it does at 1_1 or any other point of $A_1\ B_1$. From this it is evident that when the point where a wheel produces its maximum moment lies in the segment dominated by that wheel, as 2_1 in segment II, then is the moment at that point greater than that under any wheel whose point of maximum moment lies in the same segment, as W_1 at 1_1 in segment II.

It will be quite unnecessary in practice to actually subdivide the span $A_1\ B_1$ into segments as has been done. The wheel's weight whose points of maximum moment lie within the segments dominated by those wheels are the same wheel weights which are intersected in the weight line by the slope line $h_a\ h_b$. If any one of the wheel weights stands near G, the center of gravity of the set, the slope line will usually intersect that alone, but otherwise it may intersect two successive wheel weights,

and we have further to discover in this case which of the wheels produces the greater moment at its position of maximum moment.

In order to discover which of the successive wheels produces the greater moment in the span $A_1 B_1$, in case the weight line is intersected by the slope line for the span at both of these wheel weights in their position of maximum moment, let the equation for the bending moment M last used be written in the form:

$$M' = \frac{c_a}{l} L b_g - C_a c_g$$

in which equation L denotes the sum total of the wheel weights on $A B$ and C_a those on $A C$, while b_g is the distance from B to the center of gravity of L, and c_g is the distance from C to the center of gravity of C_a, both positive toward the left.

Now introduce the condition for a maximum given by equation (10), viz.: $c_a = b_g$, so that the point C where the moment M' is taken shall be as far from A as the center of gravity of L is from B. The last equation then becomes

$$M' = L \frac{c_a^2}{l} - C_a c_g,$$

in which equation, if c_a be taken as a continuous variable the point C will move in the opposite direction from the train and to an equal amount. Let the train move a distance x toward the left such that no wheel passes C.

Then the moment in the new position may be written:

$$M'' = \frac{L}{l} (c_a + x)^2 - C_a (c_g + 2x)$$

in which equation, since the moving point C has passed no one of the moving wheels, the value of C_a is unchanged.

Hence by subtraction, etc.:

$$M'' - M' = 2 \left(L \frac{c_a}{l} - C_a \right) x + L \frac{x^2}{l}.$$

Now, let the initial position of C corresponding to the moment M'' be taken in the vertical at i of Fig. 12, then the last equation reduces to

$$M'' - M' = \frac{x^2}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

for at i we have $$L \frac{c_a}{l} - C_a = 0.$$

Take M'' first at the position of maximum moment on one side of i and then at that on the other side of i.

It is evident that the last member of (11) increases equally for equal $+$ and $-$ values of x; hence $M'' - M'$ and consequently also M'' is greater at that one of the two maximum moment positions which is the more remote from i, provided no wheel leave the span in order to permit this wheel to reach its position of maximum moment.

16. To find where each wheel of a given set, all of which remain on the span, causes its maximum bending moment, in case there is a uni-

formly distributed load upon the span in addition to the set of wheel weights. Let w be the weight of the uniformly distributed load per unit of length, and then employing the same notation as before we may write the expression for the bending moment at C in the form:

$$M = L\frac{c_a\, b_g}{l} - C_a\, c_g + \tfrac{1}{2}\, w\, c_a\, (l - c_a)$$

Now let C move with the train an indefinitely small distance $\delta\, x_1$ to the left; then

$$M + \delta\, M = \frac{L}{l}(c_a - \delta\, x)\,(b_g + \delta\, x) - C_a\, c_g$$
$$+ \tfrac{1}{2}\, w\,(c_a - \delta\, x)\,(l - c_a + \delta\, x).$$

After subtracting and neglecting small terms we obtain

$$\frac{\delta\, M}{\delta\, x} = \frac{L}{l}\,(c_a - b_g) + w\,(c_a - \tfrac{1}{2}\, l),$$

an equation analogous to (10).

M reaches its maximum value when this expression vanishes. Equate it to zero and the result may be written in this form:

$$\frac{L\, b_g}{l} = \frac{L + w\, l}{l}\, c_a - \frac{w\, l}{2} \qquad \ldots\ldots\ldots\ldots\ldots (12)$$

a result which expresses the relation between c_a and b_g for a maximum moment under a wheel distant c_a from A. We say under a wheel, because it is otherwise known that a maximum can only occur under a wheel. The positions of these points of maximum moment for the successive wheels can be readily discovered as follows:

In Fig. 13 let 1′ 2′ 3′ be the positions of the wheel weights W_1 W_2 W_3 when their center of gravity is at G, the middle of the span $A_1 B_1$, which last is laid off on the same scale as the wheel distances. Let $h_a\, o = pq = L$ represent the sum total of the wheel weights, and let $om = np = \tfrac{1}{2}\, wl$ be one-half the total uniform load.

The equation of the slope line $h_a\, q$ may be written in the form:

$$y' = \frac{L}{l} b_g$$

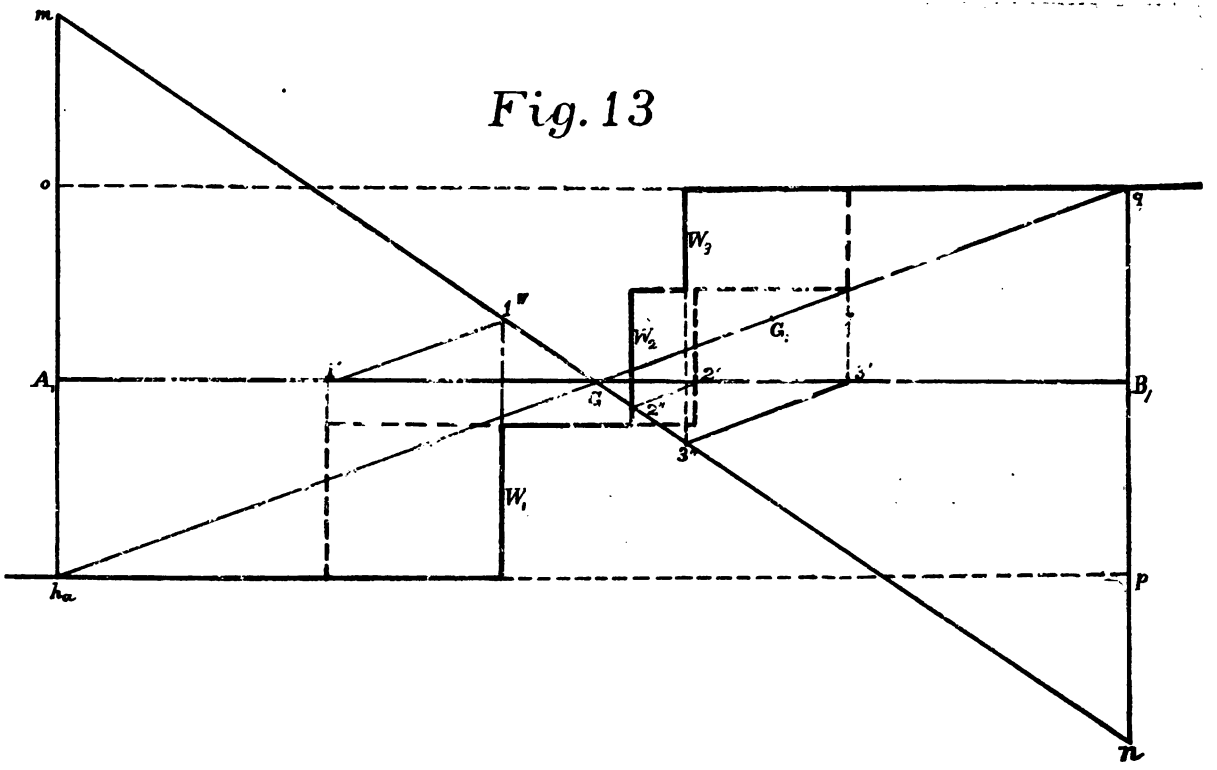
in case the origin of the co-ordinates b_g and y' is taken at q in the vertical at B, and b_g is reckoned as positive toward the left and y' as positive downwards. Again the equation of the line $m\, n$ may be written

$$y'' = \frac{L + wl}{l}\, c_a - \frac{w\, l}{2}$$

in case the origin of the co-ordinates c_a and y'' is taken at o in the vertical at A_1 and c_a is the distance from C to the vertical at A_1 and y'' is positive downwards. In order to have a maximum moment at any point C, we know that C must be taken at a wheel, and furthermore equation (12) shows that c_a and b_g must be so taken that

$$y' = y''.$$

Fig. 13



To find the point where W_1 fulfills this last condition draw $1'\ 1''$ parallel to $h_a\ q$, and suppose the set of wheels to have moved to such a position as to have moved W_1 from the vertical at $1'$ to that at $1''$; then has the center of gravity of the set also moved an equal distance and lies in the vertical at G_1. In case W_1 is at $1'$ then is y_1'' the ordinate at $1'$ between oq and mn, and y_1' is the ordinate at G_1 between oq and $h_a\ q$. When W_1 reaches $1''$ then is y_1'' the ordinate at $1''$ between oq and mn, and y_1' is the ordinate at G_1 between oq and $h_a\ q$. Our construction evidently makes $y_1' = y_1''$ in case W_1 is at $1''$, and so by equation (12) W_1 produces the greatest moment when it stands at $1''$.

Similarly it may be shown that W_2 and W_3 cause their maximum moments in the verticals at $2''$ and $3''$ respectively, provided $2'\ 2''$ and $3'\ 3''$ are parallel to $h_a\ q$. Since the slope line $h_a\ q$ intersects W_2 in its position of maximum moment, it is evident that W_2 in this position satisfies the condition given in equation (5) for a maximum moment at any point of the span, and consequently W_2 at $2''$ will produce the greatest moment in the span. But in case the slope line $h_a\ q$ should intersect two successive wheel weights in their positions of maximum moment, it will be further necessary to discover which wheel produces the greater moment of the two.

17. To find which wheel of a given set causes a greater maximum moment than any other, in case there is an additional uniform load.

As just shown, that wheel weight which is intersected by the span slope line in position of maximum moment is the one which produces the greatest maximum moment, provided there is but one such wheel. In case of several such wheels, we might proceed by the general method of Art. 12, but it is much simpler in the usual case where there are only two successive wheel weights intersected by the slope line, to proceed as in Art. 15, which will now be shown to be also correct for this kind of loading.

In the first equation of moments in the last article let the train move a distance x_g to the left, and at the same time let the point C, at which the moment is taken, move to the right a distance x_c, such that while b_g receives the increment x_g, c_a shall receive a corresponding increment, determined by equation (12), thus:

$$\frac{L}{l}(b_g + x_g) = \frac{L+w\,l}{l}(c_a + x_c) - \frac{w\,l}{2}$$

From this subtract (12) and we obtain the relation sought,

$$\frac{L}{l}x_g = \frac{L+w\,l}{l}x_c$$

Now let the train move, as proposed, a distance x_g and we have for the value of M'' the moment at C after it has moved a distance x_c

$$M'' = \frac{L}{l}(c_a + x_c)(b_g + x_g) - C_a (c_g + x_c + x_g)$$

$$+ \frac{w\, l}{2}(c_a + x_c)(l - c_a - x_c).$$

Subtract from this the moment M' at C before the movement, then

$$M'' - M' = (L\frac{c_a}{l} - C_a)\, x_g + (L\frac{b_g}{l} - C_a)\, x_c$$

$$+ \frac{L}{l} x_c x_g + \frac{1}{2} w\, l\, (l - 2c_a - x_c)\, x_c,$$

in which equation x_c and x_g have the relation previously obtained, while c_a and b_g have that given in (12). Substitute the value of b_g from (12) and we have

$$M'' - M' = (L\frac{c_a}{l} - C_a)(x_c + x_g) + L\frac{x_c x_g}{l} + \frac{1}{2} w\, x_c^2.$$

Take the initial position of C, at which the moment is M', at the point, between the positions of maximum moment of the two successive wheels, at which point

$$L\frac{c_a}{l} - C_a = 0.$$

Then our last equation becomes, after also introducing the value of x_g in terms of x_c :

$$M'' - M' = \frac{L + \frac{1}{2} w\, l}{l} x_c^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

From equation (13) it appears that $M'' - M'$, and, consequently, M'' also, is greater the greater the numerical value of x_c regardless of its sign. Consequently, of two successive wheel weights cut by the slope line while standing in their respective positions of maximum moment, that one produces the greater maximum moment whose position is further removed from i, the point where the slope line cuts the side of the weight line between them.

18. The magnitude of the maximum moments under a set of wheels all of which remain upon the span, can be expressed with great simplicity as the difference between the ordinates of a certain equilibrium polygon and a parabola, which is also an equilibrium polygon.

To show this, suppose the set of wheels $W_1 \ldots W_n \ldots W_r$, whose total number is r, to remain upon the span.

Take the bending moment under any given wheel W_n, and let W_i denote any one of the r wheel weights from W_1 to W_r inclusive.

Then, the expression for the moment may be written

$$M = \frac{c_a}{l} L\, b_g - \sum_{i=1}^{i=n-1} W_i\, c_i$$

in which b_g denotes the distance from B to G, the center of gravity of all of the wheel weights L, and the remaining letters have the significations shown in Fig. 14, all distances being taken positive toward the left.

It will be seen that we have,

$$c_i = g_i - g_n, \quad c_n = l - b_n, \quad b_g = b_i - g_i = b_n - g_n$$

By the help of these identities the bending moment may be expressed thus:

$$M = \frac{L}{l}(l - b_n)(b_n - g_n) - \sum_{1}^{n-1} W_i (g_i - g_n)$$

or

$$M = \frac{L}{l} b_n (l - b_n) - \frac{L}{l} g_n (l - b_n) - \sum_{1}^{n-1} W_i (g_i - g_n) \ldots\ldots (14)$$

If we put

$$M_2 = \frac{L}{l} b_n (l - b_n) \ldots\ldots (15)$$

$$M_1 = \frac{L}{l} g_n (l - b_n) + \sum_{1}^{n-1} W_i (g_i - g_n) \ldots\ldots (16)$$

then

$$M = M_2 - M_1 \ldots\ldots (17)$$

It is evident that M_2 is the ordinate of the moment polygon (parabola), due to a single concentrated load L as it moves across the girder, or what is the same thing, it is the moment due to a total uniformly distributed load 2 L. Hence M_2 is easily computed or constructed graphically.

We proceed to show how M_1 may also be constructed graphically, with the greatest ease (though the proof is a trifle tedious), for it will be shown that M_1 is the bending due to the concentrated loads:

$$\frac{L}{l} d_1 \ldots \frac{L}{l} d_n \ldots \frac{L}{l} d_{r-1},$$

acting at fixed points which divide the span into segments having the successive lengths:

$$\frac{l}{L} W_1 \ldots\ldots\ldots \frac{l}{L} W_n \ldots\ldots\ldots \frac{l}{L} W_r,$$

in which d_1 is the distance from W_1 to W_2, d_n that from W_n to W_{n+1}, etc.

It will be noticed that these concentrated loads are proportional to the successive wheel distances, and their sum is

$$\frac{L}{l} \sum_{1}^{r-1} d = L \frac{l'}{l}$$

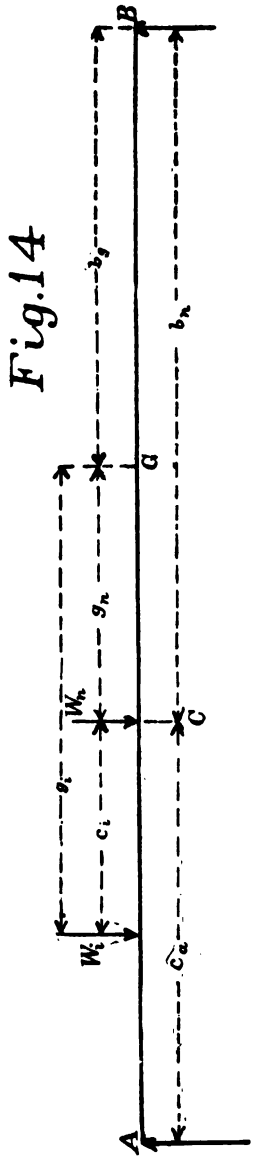


Fig. 14

in which l' is the total wheel base, and, further, that the segments of the span l are proportional to the successive wheel weights. Now, taking the moments about B of these proposed weights at their proposed positions, and calling V_1 the reaction at A, we have

$$V_1\, l = d_1 \sum_{2}^{r} W_i + \ldots + d_{n-1} \sum_{n}^{r} W_i + \ldots + d_{r-1}\, W.$$

This quantity can be expressed more simply; for, take the moments of all the actual wheel weights W_2, etc., about W_1, and we have the identity

$$L g_1 = W_2\, d_1 + W_3\, (d_2 + d_1) + \ldots + W_n \sum_{1}^{n-1} d_1 + \ldots + W_r \sum_{1}^{r-1} d_i.$$

By rearranging the terms of this equation we find it the same as the moment about B, previously obtained,

$$\therefore V_1\, l = L g_1.$$

Let M_n denote the bending moment due to the proposed concentrated weights at a point of the span distant b_n from B between the weight $\frac{L}{l}\, d_{n-1}$ and $\frac{L}{l}\, d_n$. We then find

$$M_n = V_1\, (l - b_n) - \frac{L}{l} \left[d_1 \left(l - \frac{l}{L}\, W_1 - b_n \right) \right.$$

$$\left. + d_2 \left(l - \frac{l}{L} \sum_{1}^{2} W_i - b_n \right) + \ldots + d_{n-1} \left(l - \frac{l}{L} \sum_{1}^{n-1} W_i - b_n \right) \right].$$

Substituting in this equation the value just found for $V_1 = \frac{L}{l}\, g_1$ and rearranging the terms

$$M_n = \frac{L}{l} \left[g_1 - (d_1 + \ldots + d_{n-1}) \right] (l - b_n)$$

$$+ d_1 W_1 + d_2 \sum_{1}^{2} W_i + \ldots + d_{n-1} \sum_{1}^{n-1} W_i.$$

Now $d_1\, W_1 + d_2 \sum_{1}^{2} W_i + \ldots d_{n-1} \sum_{1}^{n-1} W_i$

$$= W_1 \sum_{1}^{n-1} d_1 + W_2 \sum_{2}^{n-1} d_i + \ldots + W_{n-1} d_{n-1},$$

$$\therefore M_n = \frac{L}{l} \left(g_1 - \sum_{1}^{n-1} d_i \right) \left(l - b_n \right) + W_1 \sum_{1}^{n-1} d_i + W_1 \sum_{2}^{n-1} d_i +$$

$$\ldots + W_{n-1}\, d_{n-1}$$

But
$$d_{n-1} = g_{n-1} - g_n,$$
$$\sum_{2}^{n-1} d_i = g_2 - g_n$$
$$\sum_{1}^{n-1} d_i = g_1 - g_n.$$

By help of these equations we have—

$$M_n = \frac{L}{l} g_n (l - b_n) + W_1 (g_1 - g_n) + W_2 (g_2 - g_n) + \ldots + W_{n-1} (g_{n-1} - g_n)$$

or,

$$M_n = \frac{L}{l} g_n (l - b_n) + \sum_{1}^{n-1} W_i (g_i - g_n)$$

from which it appears that $M_n = M_1$, *i. e.*, the bending moment M_1 may be constructed as that arising from the proposed concentrated weights applied at the points proposed.

This proposition respecting M_1 was first demonstrated by the author about 1885, and was communicated by him to Professor Maurice Levy, who published it with due credit, but demonstrated differently, in his great work *La Statique Graphique, IV^e partie*, p. 334.

19. The foregoing development is important, as giving in a single graphical construction the maximum moments under a set of wheels throughout the entire span, but is still more important, perhaps, as leading to a new formula for an approximate value of an equivalent uniform load, which formula we shall now deduce.

It has just been shown in equation (17), that the part M_2 of the total bending moment is the same as that which would be caused by a uniformly distributed load of total amount $2L$, *i. e.*, the equivalent charge per unit of the span is

$$w'' = \frac{2L}{l}$$

The other part M_1 of the bending moment may be regarded as due to concentrated loads $\frac{L}{l} d_1$, etc. In case there are three or more of these at distances not excessively unequal, the moment polygon for M_1 differs but little from a parabola. It may in all ordinary cases be taken to coincide very approximately with the equilibrium polygon (parabola) of the sum of these proposed concentrated weights taken as uniformly distributed. Their sum is

$$\frac{L}{l} \sum_{1}^{n-1} d_i = \frac{L}{l} l'$$

in which $\sum_{1}^{n-1} d_1 = l'$ is the length of the load or wheel base. The charge

per unit of length of this load when uniformly distributed is

$$w' = \frac{L}{l^2} l'$$

Hence the total equivalent uniform load is approximately

$$w = w'' - w' = \frac{2L}{l}\left(1 - \frac{l'}{2l}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

It is to be noticed that the last term in (18) takes account of the degree of concentration of the set of wheel weights, and it varies from the value zero, in case the set consists of a single wheel (*i. e.*, $l' = 0$) to the value $\frac{1}{2}$ in case the wheel base covers the entire span, with intermediate values when l' is less than l. It would seem most proper to use (18) in the following manner, in order to apply it to one or more locomotives in the middle of an unlimited uniform train. Let E be the *engine excess*, *i. e.*, the amount by which the locomotives all together exceed the weight of an equal length of the uniform train. We can then consider the span to be loaded throughout with the uniform unlimited train causing a charge of w_t per unit of length, and to that must be added the charge due to a total engine excess, E occupying a length l' supposed to be less than l.

The bending moments in the span will then be due to a charge per foot of

$$w = w_t + \frac{E}{l}\left(1 - \frac{l'}{2l}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

Objection may be made to taking the uniform train both in front of the engines and behind the locomotives. But in computing bending moments it is not objectionable so to take the loading, for it is on the side of safety and might sometime actually occur. The length l' should be taken as the total distance between the parts of the uniform train before and behind the locomotives.

20. We shall conclude the discussion respecting the stress due to a limited set of wheels, all of which remain on the span, by finding which wheel produces the maximum shear at any point of a simple girder and then deriving a convenient expression for the weight of a single load which, in conjunction with a uniform distribution, will cause a safe approximation to the maximum shear due to the actual concentrations.

In order to compare together the shearing stresses just at the left of the several wheel-weights, consider a set of r wheels $W_1 \ldots W_n \ldots W_r$, of which W_n is any one, all of which remain on the span. Let $L = \sum_1^r W$ denote the total load on the span AB. Then the shear just at the left of any wheel W_n may be written in the form

$$S = \frac{L}{l} b_g - \sum_1^{n-1} W$$

in which equation the summation includes all the wheels at the left of W_n. The notation will be understood from Fig. 14, where we have

$$b_g = b_n - g_n$$

Substituting this value of b_g in the previous equation, we may write the shear thus:

$$S = \frac{L}{l}\, b_n - \left(\frac{L}{l}\, g_n + \sum_{1}^{n-1} W\right) \ldots\ldots\ldots\ldots (20)$$

In this equation the shear has been separated into two parts, which can conveniently be separately considered.

Write equation (20),

$$S = S_2 - S_1,$$

then the part

$$S_2 = \frac{L}{l}\, b_n,$$

depends solely upon the abscissa b_n of the point in the span where we take the shear, and it increases uniformly from zero at B to L at A.

In Fig. 15 let $A\,B = l$ and $B\,B_o = L$, then $A\,B_o$ has the slope $L \div l$. Through the center of gravity G draw $A_o\,B$, at the same slope, then is $A_o\,A' = L$, and the vertical ordinate $m_1\,n_1$ between these lines at any point has the value S_2 for that point, as shown in Fig. 15 at W_1. S_2 would have the same value at this point of the span whatever wheel may stand at the point. The other part of S is

$$S_1 = \frac{L}{l}\, g_n + \sum_{1}^{n-1} W,$$

and depends only on the number and weight of the wheels at the left of W_n, and not at all upon the position of the set upon the span.

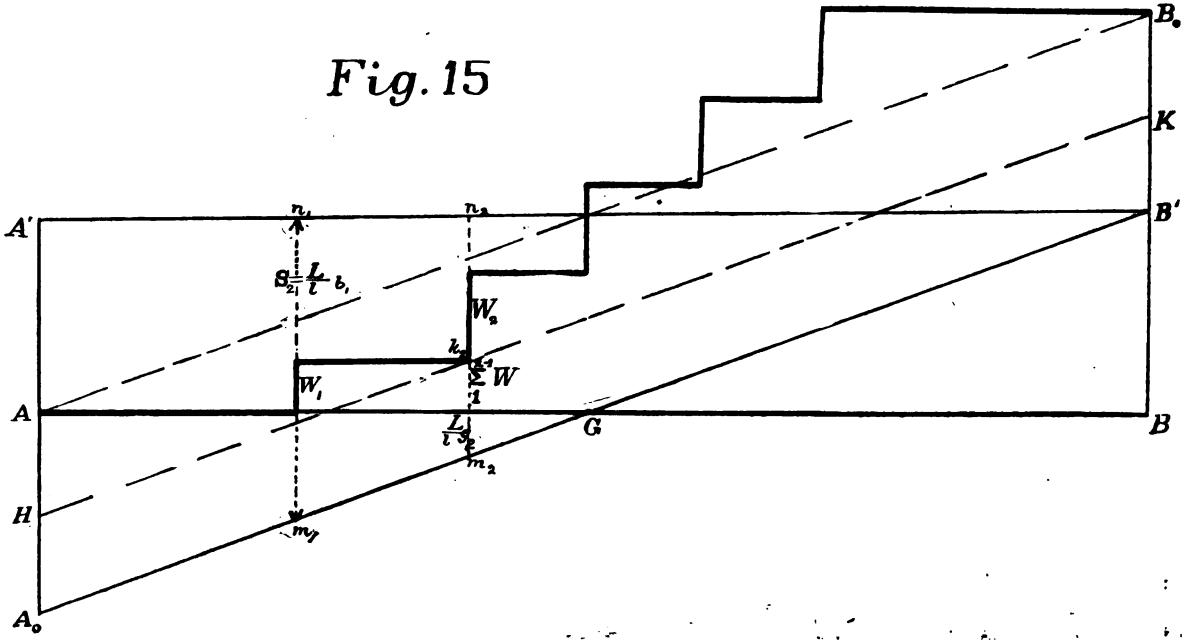
The first term of S_1 is the ordinate between $A_o\,G$ and $A\,G$, and the second term is the ordinate at any wheel between $A\,G$ and the lower extremity of the wheel weight, W_n. Now G moves with the set, hence g_n is constant. There is, therefore, a constant value, S_1, for each wheel of the set, regardless of its position upon the span.

In order that S may have its greatest value at a given point of the span S_1 must have its smallest value. To find the least value of S_1, draw $H\,K$ at the slope $L \div l$ in the lowest position it can be put, and touch the weight line at any point, as at k_2. The vertical ordinate $m_2\,k_2$ between $A_o\,B'$ and $H\,K$ is the smallest possible value of S_1, and the wheel weight which it touches is the one which causes a greater shear than any other so long as all the wheels remain on the span.

$$\therefore S_{max} = m_2\,n_2 - m_2\,k_2 = k_2\,n_2.$$

It has been shown, therefore, that the maximum value of the shear at any point of the span $A'\,B'$ is simply the vertical ordinate at that point between $H\,K$ and $A'\,B'$.

Fig. 15


B₀
K
A'
B'
n₁
n₂
$S_2 \frac{L}{l} b_1$
W_2
k_2
$\sum_1^{2-1} W$
W_1
A
G
B
$\frac{L}{l} S_2$
m_2
H
m_1
A_0

21. It is evident that the method just developed, of finding the greatest shear due to a limited set of wheels on the span, may be applied to one or more locomotives followed by a uniform train, in either of two ways. Either add the greatest shear due to the locomotives to that due to the train, or add that due to the locomotive excess to that due to a uniform train whose head is taken to be at W_1.

It will be unnecessary here to discuss that part of the total shear which is due to a train extending without limit toward the right, further than to remark that its amount at any point to the left of the head of the uniform train is

$$S_w = \frac{w\, b^2}{2\, l}$$

in which equation b is the distance from B to the head of the uniform train of which w is the weight per unit of length. S_w added to the shear obtained in equation (20) gives as the expression for the total shear just at the left of W_n, and due both to the locomotives and train:

$$S = \frac{L}{l}\, b_n - \frac{L}{l}\, g_n - \sum_1^{n-1} W + \frac{w b^2}{2\, l} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

in which equation it is understood that all the locomotive wheels are upon the span, and L denotes the weight of the locomotives alone, and b is the length of that part of the uniform train between B and the locomotives.

Again: let us take a different interpretation of (21). Let E denote the engine excess, and let it be superposed upon a uniform train extending from B to W_1 at the head of the locomotives. We may, then, for long spans write as an approximate value of the shear at the left of W_1:

$$S = \frac{b_1}{l}\left(E + \frac{1}{2}\, w b_1\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

which will give a shear somewhat in excess of the actual maximum shear for any span long enough to take in all the locomotive wheels. Equation (22) takes the whole engine excess to be situated at W_1 and disregards the second term of (21). The term thus neglected is more than enough to make up for the error committed in taking the shear at the left of W_1 throughout the entire span, instead of taking it just at the left of W_2 or W_3 in the segments dominated by those wheels.

In spans shorter than the wheel base of the locomotives it is evident that the total engine excess is in general less than in the case just considered, though the reverse might occur with a light tender and heavy uniform load, when the tender was not upon the span.

Nevertheless, equation (22) can be used as a useful and safe approximation for short spans as well as long ones.

22. It has been shown in course of the discussion of the properties of the reaction polygon in connection with Fig. 2, that the numerical

value of the shearing stress in a girder, or the vertical component of the stress acting in the web members of a single intersection truss, may be found at any point of a given span by simply reading to scale the vertical ordinate at the point between the line of zero shears for the span and the weight line in case of a girder, or between the zero line for the span and the stepped line of zero shears for the panels in case of a truss.

It is, moreover, well known that the numerical value of the bending moment at any point of a girder may be obtained by simply reading to scale the vertical ordinate at the point between the closing line for the span and the equilibrium polygon due to the weights. Such an equilibrium or moment polygon is drawn upon the large diagram, Fig. 3.

The discussion of the moment polygon is a matter somewhat aside from the main purpose of this paper, yet its employment so greatly facilitates practical work that we here give it a brief consideration.

It may seem to the practical bridge designer that the measurement of moments upon such a diagram cannot be made with sufficient accuracy for bridge calculation, but, in fact, perfectly satisfactory readings are now made in the office of the Chief Engineer of the C. N. & T. P. Railway, where this method of reading diagram to scale has displaced that of using tables of moments, which had been employed for years. Of the great convenience and rapidity of the graphical method there can be no question.

A method of constructing the moment polygon and moment lines for each separate wheel weight on profile paper to the scale on which the paper is itself ruled, in such a manner as to avoid all inaccuracies due to drawing parallels, as well as all uncertainties as to scale, and so make the diagram equivalent to a table of moments, has been published by Ward Baldwin, C.E.* This superior method is shown in Fig. 3.

23. To construct and use the moment polygon 1, 2, 3, etc., in Fig. 3, proceed as follows: Take the horizontal line 0 0, which is the zero line from which loads are read, to be also the zero line for moments. At one hundred feet (100′) to the right of wheel 1, lay off from 0 0 a vertical ordinate $= W_1$ pounds. Through its upper extremity and the foot of W_1 draw the moment line 11.

If we take the ordinate at 100′ to represent 100 W_1 foot-pounds, then the scale of moments in foot-pounds is one one-hundredth the scale of weights in pounds, and the ordinate between 0 0 and 11, at any point, is evidently the moment of W_1 about that point in foot-pounds.

Again, at 100′ from W_2 lay off from 11 a vertical ordinate $= W_2$ pounds. Through its upper extremity and the point where 11 intersects the vertical at W_2, draw the moment line 22. Then is the ordinate between 11 and 22, at any point, the moment of W_2 about that point; and the ordinate between 0 0 and 22 at any point is the moment of W_1 and

* *Engineering News*, September 28th, 1889.

W_2 about that point. Draw in a similar manner the moment lines 3 3, 4 4, etc.

The uniform train is taken as concentrated in sections of 10′ each at verticals 19, 20, etc., as also at 0, — 1, — 2, etc. These last concentrations give the moment lines — 1 — 1, — 2 — 2, etc. It will be easy to test the accuracy of the construction by computing the moments of the successive wheels about certain fixed points, say at 20′ or 50′ from W_1, and see whether the successive moment lines cross the ordinate at the fixed point at their proper heights. There need be no difficulty in making these constructions to such scale and of such proved accuracy at every point as to merit the full confidence of the practical designer.

These moment lines and the moment polygon formed by them can be employed in several ways, among which we may notice:

First.—The bending moment at any point of a girder is the vertical reading between the polygon and the closing line or thread stretched between those points of the polygon which are in the verticals at the extremities of the span. A scale consisting of a piece cut from the same ruled sheet will be found convenient.

Second.—The bending moment at any pin of the unloaded chord of a single intersection truss is the vertical reading, at that pin, between the closing line for the span and the closing line for the panel.

Third.—The moment of any number of wheels to the left of any point about that point is the reading on the vertical at the point between the moment lines for those wheels. In this manner the value of $\sum_{A}^{B} Wb$ may be read and its value used in computing the shear at any point or in finding center of gravity.

Fourth.—It will be possible to find by actual trial in many cases which one of the wheels produces the greatest bending moment at a given point of the span, especially in case of short spans, by finding at which wheel the ordinate is greatest between the closing line and the moment polygon. In such trials upon the moment polygon a hollow right-angled triangle cut from the same ruled sheet as the diagram will be found to be a great convenience, as also for use in similar trials upon the reaction polygons; for, taking a point upon its horizontal edge as A such that B falls upon the vertical edge, we readily locate the extremities of the span so as to fulfill one condition or another.

24. It is also possible by help of the moment polygon to find in a simple manner those segments of a girder resting on end supports in which the maximum shear is dominated by the successive wheels near the head of the train. This method of solution is due to Ward Baldwin, C. E.,* and is as simple, graphically, as that already given by help of reaction polygons. It is as follows:

* *Engineering News*, October 12th, 1889.

Take the expressions already given in Art. 6 for S' and S'' the shears just at the left of C' and C'' respectively, in which the last terms are the total loads to the left of these points. Let C' and C'' be situated at the wheels at which the shears are to be compared. Then the final summations in the expressions for S' and S'' will not include the wheel weights at C' and C'' respectively.

In order to find the point where two wheels produce the same shear, let $S' = S''$, and equating their values as given in Art. 6, transposing, canceling, etc., we have

$$\sum_{A''}^{B''} W b'' - \sum_{A'}^{B'} W b' = l \sum_{C'}^{C''} W \ldots\ldots\ldots\ldots \quad (23)$$

in which $b'' - b'$ for any wheel is the distance which it is necessary to move the train to bring one of the wheels considered to the position previously occupied by the other.

The first term in equation (23) is the height of the moment polygon above 0 0, the zero line of moments at the vertical through B''. The second term is the height of the moment polygon above 0 0 at the vertical through B'. The distance between the verticals at B'' and B' is $b'' - b' = d$, the distance apart of the wheels under which the shears are taken as equal. Hence the first member of equation (23) is the difference in height of two points on the moment polygon whose horizontal distance is d.

The second member shows how much this vertical height must be, for it is the product of the weight of each wheel between C'' and C' multiplied by l, and the product for each wheel can be read off upon the diagram at a distance l to the right of each of those wheels.

Hence to find the point in the span where W_1 and W_2 produce the same shear, read off $l\ W_1$ on the diagram and lay it off downward from the upper left corner O of a rectangular card or separate slip of the ruled sheet, call it Om. Lay off horizontally from the same corner $On = d$, then are Om and On the vertical and horizontal distances between the points of the moment curve at B' and B''. Place the card so that m falls upon the moment polygon, and slide m along the polygon, keeping the card horizontal until n also is upon the moment polygon.

Then m is in the vertical at B' and n in that at B'', such that the shear under W_1 is the same when the right end of the span is at B'. as it is under W_2 when the right end of the span is at B''. The segment dominated by W_1 is, therefore, the distance from W_1 to B', or from W_2 to B'', in the case when this condition is fulfilled. The point where W_2 W_3 produce the same shear may be found in similar manner.

This demonstation supposes that no wheels pass off the span, for had some wheels left the span then some other line than 0 0 would be the line of zero moment.

PART II.

MAXIMUM STRESSES UNDER CONCENTRATED LOADS IN CANTILEVERS WITH TWO SUPPORTS; AND IN DOUBLE INTERSECTION TRUSSES WITH END SUPPORTS.

1. The cantilever $E\,D$, Fig. 1, is supposed to be continuous between its extremities E and D and to rest upon supports at A and B, while the spans $F\,E$ and $D\,G$ are supported at their extremities F and G respectively by piers, but at their other extremities, D and E, they are supported by the cantilever itself.

It will be convenient in this discussion of the cantilever to employ a systematic notation like that previously employed, viz.:

Let a denote the distance of any wheel from A, b its distance from B, c from C, etc., etc., and let the distance a be taken as positive when the wheel is at the left of A, but negative when it is situated to the right of A. Similarly, let the distances of a wheel from any other points, as B, C, etc., be taken to be positive toward the left. Also let l_a denote the span $E\,A$ at the left of A, and L_a the total load on the span l_a, and similarly for the other spans at the left of the points B, D and E, with their loads as shown in Fig. 1. Then shall we have the identities:

$$b - a = l_b, \quad d - b = l_d, \text{ etc., etc.}$$
$$d - a = l_b + l_d, \quad e - d = l_a + l_b + l_d, \text{ etc., etc.}$$
$$c - a = c_a, \quad c - e = l_a + c_a, \text{ etc., etc.}$$

2. The position of the train when it produces the numerically greatest shear at any point C of the end span $B\,D$ of the cantilever $E\,D$, in Fig. 1, may be investigated in a manner similar to that previously employed with the simple girder, thus:

It is well known, and need not be here demonstrated, that the greatest shear at any point C in $B\,D$ will be under one of the wheels near the head of the train when the train extends from G to near C, the point considered. In particular the case need not be investigated of a train extending from any point in the opposite direction as F or A and extending up to C; or what is the same thing, it is unnecessary to investigate the shear at points of $E\,A$ when the train extends from A or G or from any point between A and G up to or near the point of $E\,A$ under consideration; for the shear in $B\,D$ is greatest when the train approaches from G, and greatest in $E\,A$ when it approaches from F.

Let V_d be that part of L_g which is supported by the extremity D of the cantilever, then the shear S just at the left of any point C of $B\,D$ is

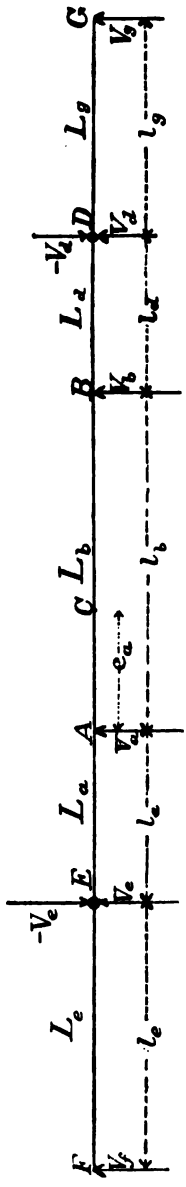


Fig. 1

$$S = V_d + \sum_{C}^{D} W. \quad \text{But} \quad V_d\, l_g = \sum_{D}^{G} Wg,$$

$$\text{hence,} \quad S = \sum_{D}^{G} W \frac{g}{l_g} + \sum_{C}^{D} W,$$

in which equation the sign of S is taken as positive when produced by a force acting upward upon the cantilever at the left of C or downward at the right of C, and *vice versa*.

In order to compare the shear under one wheel standing at C with that under some other wheel at a distance x to the right of it when it also arrives at C, let S' denote the shear under the first wheel when at C, and S'' that under the other wheel when at C. Let the train be stationary, and suppose the cantilever to be shifted toward the right a distance x under the train from its initial position, in which C is at C', to the final position in which C is at C''. Denote the corresponding positions of D by D' and D'', and so on for the points A, B, etc.

We then obtain by subtraction :

$$S'' - S' = \left[\sum_{D'}^{G'} - \sum_{D'}^{D''} + \sum_{G'}^{G''}\right] W \frac{g''}{l_g} - \sum_{D'}^{G'} W \frac{(g'' - x)}{l_g}$$

$$+ \left[\sum_{C'}^{D'} - \sum_{C'}^{C''} + \sum_{D'}^{D''}\right] W - \sum_{C'}^{D'} W,$$

in which equation the summations in the first square bracket are together equivalent to a single summation from D'' to G'', and those in the second to one from C'' to D'' by reason of the identities :

$$D''\ G'' = D'\ G' - D'\ D'' + G'\ G''$$
$$C''\ D'' = C'\ D' - C'\ C'' + D'\ D''$$

Also $g' = g'' - x$.

Cancel and make use of the equality $g'' - d'' = l_g$, and we have :

$$\frac{S'' - S'}{x} = \frac{1}{l_g}\left[\sum_{D'}^{G'} W + \sum_{G'}^{G''} W \frac{g''}{x} - \sum_{D'}^{D''} W \frac{d''}{x}\right] - \frac{1}{x} \sum_{C'}^{C''} W \ldots . \quad (1)$$

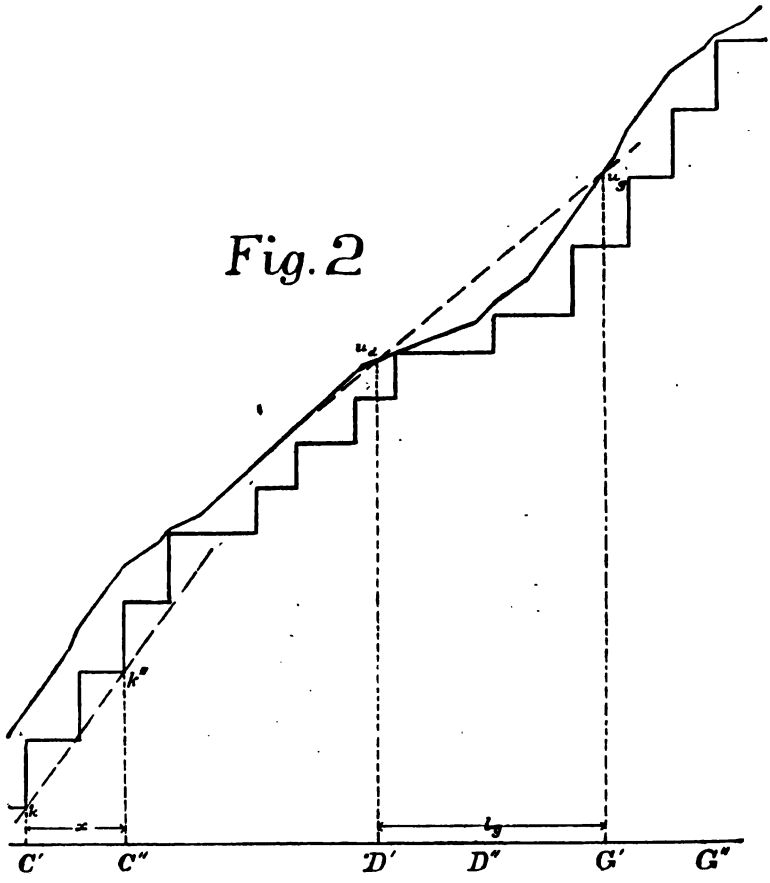
an equation which admits of a simple geometrical interpretation as shown in Fig. 2.

Let u_d and u_g lie on the reaction polygon for the span whose length is x ; then the first group term in the last member of equation (1) is the tangent of the inclination of the slope line $u_d\ u_g$ to the horizontal, and the last term of equation (1) is the tangent of the inclination of the slope line $k'\ k''$ for the load in the span $C'\ C''$.

When the slope of $u_d\ u_g$ is greater than that of $k'\ k''$, then is $S'' > S'$, and *vice versa*.

It is to be noticed that the slope of $u_d\ u_g$ will not greatly vary as an ordinary train moves across the span, especially when l_g is large, and

Fig. 2



consequently one and the same wheel near the head of the train will usually dominate the shear in the whole of $B\ D$.

3. To discover what position of the train produces the greatest shear at any point C of the middle span $A\ B$ of the cantilever $E\ D$, Fig. 1, the procedure is similar to that just detailed: The shear S at C is given by the equation

$$S = V_a - V_e - \sum_{E}^{C} W$$

By taking moments about B we find in notation already explained

$$V_a\ l_b = \sum_{E}^{D} W\ b - V_d\ l_d + V_e\ (l_a + l_b)$$

Now,

$$V_d\, l_g = \sum_{D}^{G} W g,$$

and

$$V_e\, l_e = -\sum_{F}^{E} W f.$$

Hence we find by substitution

$$S = \left[\frac{1}{l_b}\sum_{E}^{D} W b - \frac{l_d}{l_g}\sum_{D}^{G} W g - \frac{l_a}{l_e}\sum_{F}^{E} W f\right] - \sum_{E}^{C} W.$$

It is known that the shear is greatest just at the left of some wheel.

In order to compare the shears under (*i. e.*, just at the left of) two wheels at a distance x from each other, let C' and C'' denote their respective positions of the two wheels on the load line. Then using notation previously employed, we find

$$\begin{aligned} S'' - S' = &\left[\sum_{E'}^{D'} + \sum_{D'}^{D''} - \sum_{E''}^{E''}\right] W\frac{b''}{l_b} - \sum_{E'}^{D'} W\frac{(b''-x)}{l_b} \\ &- \frac{l_d}{l_g}\left[\sum_{D'}^{G'} + \sum_{G'}^{G''} - \sum_{D'}^{D''}\right] W\frac{g''}{l_b} + \frac{l_d}{l_g}\sum_{D'}^{G'} W\frac{(g''-x)}{l_b} \\ &- \frac{l_a}{l_e}\left[\sum_{F''}^{E''} + \sum_{E'}^{E''} - \sum_{F''}^{F''}\right] W\frac{f''}{l_b} + \frac{l_a}{l_e}\sum_{F'}^{E'} W\frac{(f''-x)}{l_b} \\ &- \left[\sum_{E'}^{C'} + \sum_{C'}^{C''} - \sum_{E'}^{E''}\right] W + \sum_{E'}^{C} W. \end{aligned}$$

Cancel and notice that $b'' - a'' = l_b$, and the equation may then be written in the form,

$$\begin{aligned} \frac{S''-S'}{x} = &\frac{1}{l_b}\left[\sum_{E'}^{D'} W + \sum_{D'}^{D''} W\frac{b''}{x} - \sum_{E'}^{E''} W\frac{a''}{x}\right] \\ &- \frac{l_d}{l_b\, l_g}\left[\sum_{D'}^{G'} W + \sum_{G'}^{G''} W\frac{g''}{x} - \sum_{D'}^{D''} W\frac{g''}{x}\right] \\ &- \frac{l_a}{l_b\, l_e}\left[\sum_{F'}^{E'} W + \sum_{E'}^{E''} W\frac{f''}{x} - \sum_{F'}^{F''} W\frac{f''}{x}\right] - \frac{1}{x}\sum_{C'}^{C''} W. \end{aligned}$$

In the first bracket put

$$b'' = d'' - l_d, \text{ and } a'' = e'' + l_a;$$

in the last term of the second put $g'' = d'' + l_g$; and in the middle term of the third put $f'' = e'' - l_e$. Cancel, and the equation becomes

$$\frac{S''-S'}{x_1}=\frac{1}{l_b}\left[\sum_{E'}^{D'}W+\sum_{D'}^{D''}W\frac{d''}{x}-\sum_{E'}^{E''}W\frac{e''}{x}\right]$$

$$-\frac{1}{l_b}\left[\sum_{D'}^{G'}W+\sum_{G'}^{G''}W\frac{g''}{x}-\sum_{D'}^{D''}W\frac{d''}{x}\right]\frac{l_d}{l_g}$$

$$-\frac{1}{l_b}\left[\sum_{F'}^{E'}W+\sum_{E'}^{E''}W\frac{e''}{x}-\sum_{F'}^{F''}W\frac{f''}{x}\right]\frac{l_a}{l_e}-\frac{1}{x}\sum_{C'}^{C''}W\ldots(2)$$

an equation which may be interpreted as shown in Fig. 3.

Let $u\ u$ be the reaction polygon for the span whose length is x; then the first bracket of the last member in (2) expresses the height of u_d above u_e, the second the height of u_g above u_d, and the third the height of u_e above u_f. It is evident, from similarity of triangles, that the second bracket multiplied by $l_d \div l_g$ expresses the height of u_d above t_b; and the product of the third bracket by $l_a \div l_e$ expresses the height of t_a above u_e: hence the total coefficient of $1 \div l_b$ expresses the height of t_b above t_a. This height divided by l_b is the tangent of the angle which the slope line $t_a\ t_b$ makes with the horizontal.

Again, the coefficient of $1 \div x$ in the last term of (2) is the sum total of the wheel weights which pass C during the movement x, including that standing at C' and excluding that at C'', and this divided by x is the tangent of the angle which $k'\ k''$ makes with the horizontal. When the slope of $t_a\ t_b$ is the greater, then is $S'' > S'$, and *vice versa*.

It will not be difficult to determine the segments of $A\ B$ in which the greatest shears are dominated by the several wheels near the head of the train, for these wheels cause greater shears than those more distant from the head of the train, as appears at once from the general equation for S given above.

First find the point of $A\ B$ where S_1 the shear just at the left of W_1 is equal to S_2, the shear just at the left of W_2, on the supposition that W_1 does not leave $A\ B$ when W_2 arrives at the point mentioned, as it usually will not. Since the train at most extends from G to some point between A and B, there are no wheels between F and A, so the third bracket vanishes as well as the third term of the first bracket in (2). Hence, in this case, the reaction polygon between u_f and u_a coincides with the horizontal through t_a, as does also the slope line $u_f\ t_a$, as well as the weight line. See Fig. 4.

Now draw $t_a\ t_b$ parallel to $k_1\ k_2$, then the height of t_b above t_a becomes known $= B'\ t_b$.

Now let a card be cut in the shape represented by the broken lines, such that three parallel vertical edges are at the respective distances apart of the verticals at B, D and G. Place the left hand upper corner of the card at the height of t_b above t_a, then shift the card horizontally (so

Fig. 3

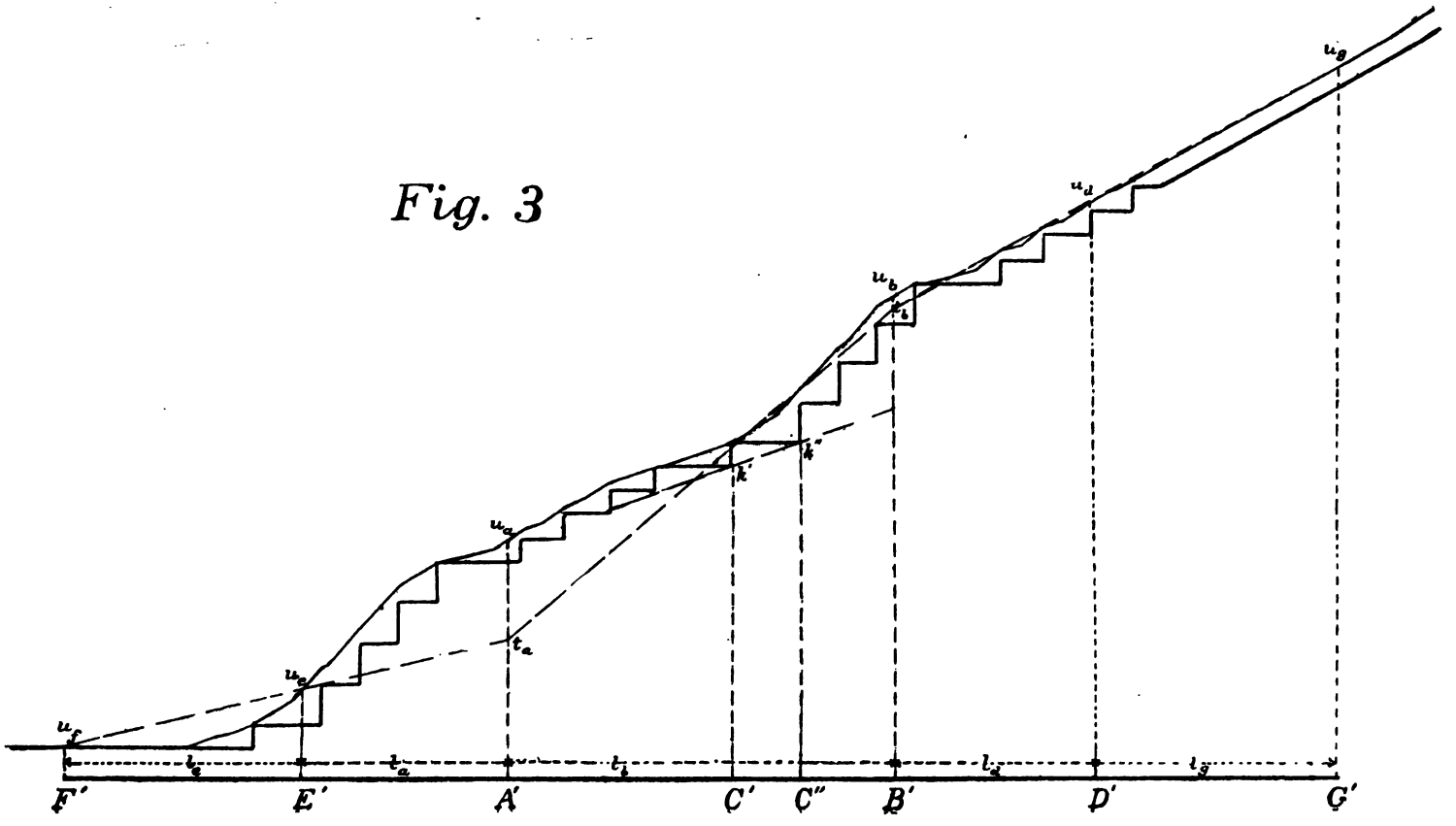
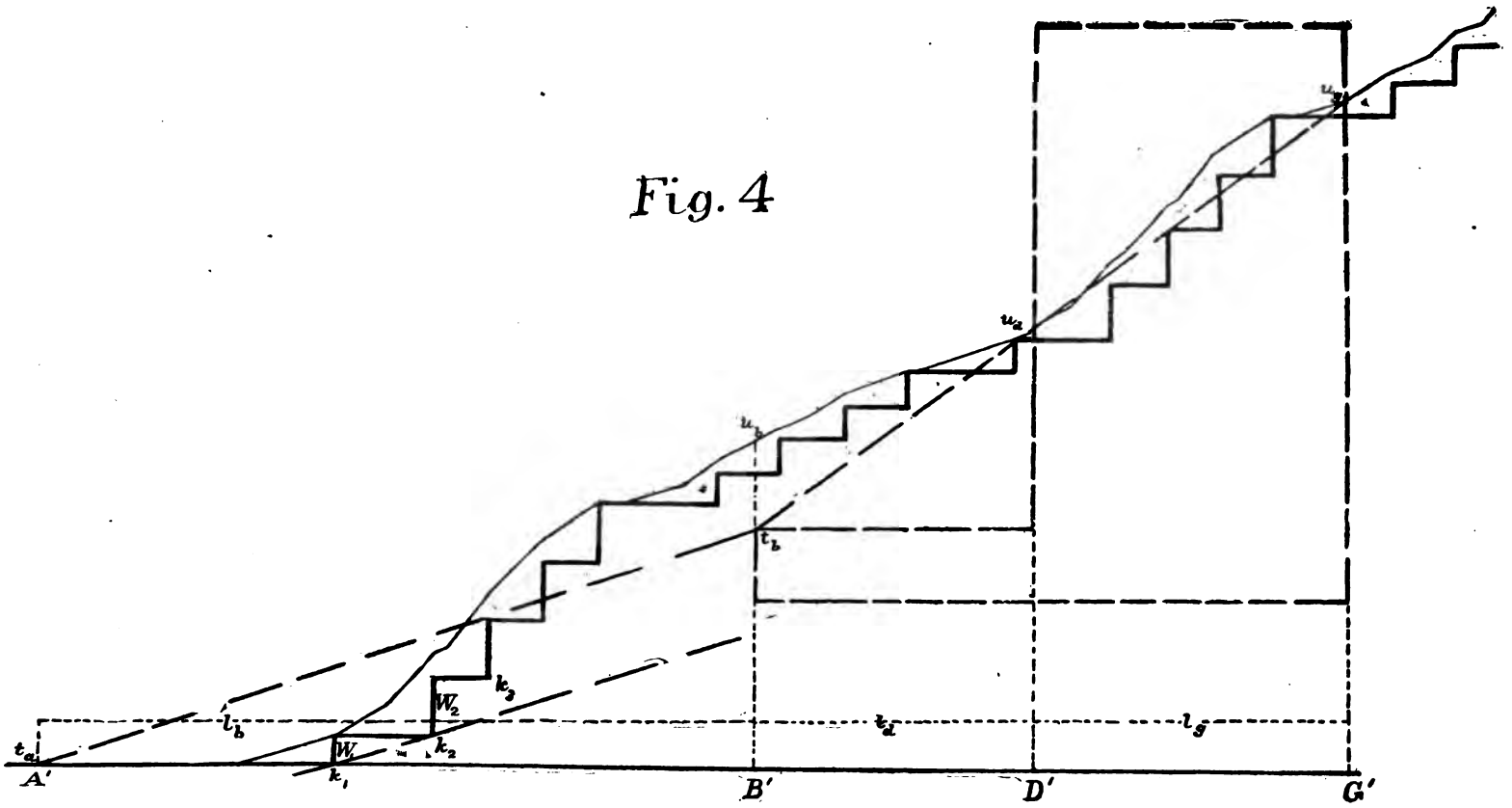


Fig. 4



that this left upper corner will remain at the same height) until the points u_d and u_g, where the other two vertical edges of the card cut the reaction polygon, fall in the same straight line with the upper left corner. This gives the position of the train, or better, the position of the cantilever when the head of the train has arrived at a point where the shear just at the left of W_1 is the same as the shear just at the left of W_2 will be, when W_2 arrives at that point. The greatest shear in that segment of $A\,B$ at the right of the point so determined is dominated by W_1; and at the left of this point the shear is greatest just at the left of W_2 until W_2 arrives at the point where the shear just at the left of W_2 is equal to that which will exist just at the left of W_2 when W_3 shall arrive at the same point. The construction for finding this second point will be similar to that for the first point, but in this case $t_a\,t_b$ must be drawn parallel to $k_2\,k_3$ instead of $k_1\,k_2$. It will seldom be necessary to find this second point, as the whole of $A\,B$ will be practically dominated either by W_1 or W_2.

Moreover, from the general expression for S given above it not only appears that the shear is greatest near the head of the train, *i. e.*, when the train extends not far to the left of C, but it also appears clear that the shear again begins to increase after the head of the train passes to the left of A, as is evident from the fact that $b > l_b$ for those wheels at the left of A. And since f is really negative in $F\,E$, the shear at C will generally increase until the head of the train reaches F.

A comparison by the method of Fig. 3 will show whether the shear at C is greatest when the head of the train is near C or near F.

The general expression for S also shows that the shear is greater, other things being equal, in case the train is so short that no wheels stand between B and G. The length of the wheel base for the locomotive may be such that this condition cannot be exactly fulfilled. Nevertheless, it should be assumed that the train consists of one locomotive and tender, or two locomotives and tenders with or without uniform train, according as the one or the other assumed loading will cause the greater shear at C. The assumed uniform train should be taken as not extending to the right of B in case that assumption tends to increase the shear.

If no loading is regarded as resting upon that part of the cantilever at the right of B, and it is desired to compare by the method of Fig. 3, the shear at C in case the head of the train is near C with the shear at C in case the head of the train is near F, then it will be necessary to consider B as a free end supported on a pier at B without constraint, or what is the same thing, take l_d and l_g as of infinitesimal length, so that D and G may be for the moment regarded as coinciding with B, though in reality they do not so coincide. This will make t_b coincide with u_b in Fig. 3.

It is to be noticed that in case the greatest shear occurs with the head

of the train near C, and there is no load to the right of B, this differs not at all from the case of the greatest shear at C in a simple span $A\ B$, as given in the previous paper. Having now discussed briefly the greatest shear in a simple girder cantilever, we shall proceed to treat the single intersection truss cantilever.

4. To discover the position of the cantilever $E\ D$, Fig. 1, with reference to the train when the shear is greatest in the web members of a given panel $H\ K$, in case the cantilever is a single intersection truss with parallel chords.

1°. In the end span $B\ D$ at any point C of the panel $H\ K$, the shear in the truss may be written—

$$S=\sum_{D}^{G} W\frac{g}{l_d}+\sum_{C}^{D} W-\sum_{H}^{K} W\frac{k}{p}+\sum_{H}^{C} W$$

in which equation p is the length of the panel $H\ K$, and k is the distance of any wheel to the left of its right extremity.

Let the train be supposed to move an indefinitely small distance, $\delta\ x$ toward the left, then—

$$S+\delta\ S=\sum_{D}^{G} W\frac{(g+\delta\ x)}{l_d}+\sum_{H}^{D} W-\sum_{H}^{K} W\frac{(k+\delta\ x)}{p}$$

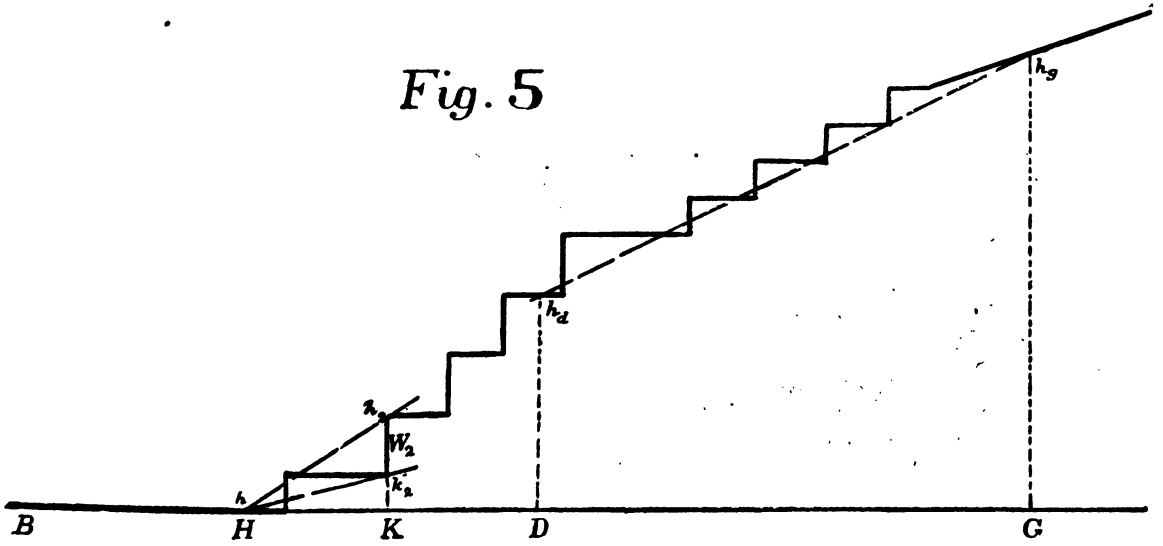
Hence

$$\frac{\delta\ S}{\delta\ x}=\frac{1}{l_d}\sum_{D}^{G} W-\frac{1}{p}\sum_{H}^{K} W \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

From equation (3) it appears that when the head of the train, as it moves from G toward B, has entered upon the panel $H\ K$ so far that some one of the wheels has arrived at K, which changes the sign of the last member of (3) from positive to negative as it passes K, then will S reach a maximum which is also its numerically greatest value. The first term in the last member of (3) is the tangent of the slope of the line $h_d\ h_g$ in Fig. 5, while the last term is that of the slope line in $H\ K$, *i. e.*, either $h\ h_2$ or $h\ k_2$, according as W_2 stands or does not stand in the panel. Hence it is evident that maximum value of S occurs in $H\ K$ when W_2 stands at K. The general slope of $h_d\ h_g$ will not greatly vary during the passage of the train; hence the same wheel will usually command the greatest shear in every panel of $B\ D$.

2°. In the middle span $A\ B$, at any point C of the panel $H\ K$, the shear in the truss may be written as the difference between the shear in the cantilver regarded as a girder and the shear borne by the stringer

Fig. 5



$H K$, compare Art. 3. Hence—

$$S = \frac{1}{l_b}\left[\sum_{E}^{D} W b - \frac{l_d}{l_g}\sum_{D}^{G} W g - \frac{l_a}{l_e}\sum_{F}^{E} W f\right] - \sum_{E}^{C} W$$

$$- \frac{1}{p}\sum_{H}^{K} W k + \sum_{H}^{C} W,$$

in which equation k is the distance of any wheel to the left of the right panel point K.

Now, if the train move an indefinitely small distance δx toward the left, we obtain in the same manner as before—

$$\frac{\delta S}{\delta x} = \frac{1}{l_b}\left[\sum_{E}^{D} W - \frac{l_a}{l_g}\sum_{D}^{G} W - \frac{l_a}{l_e}\sum_{F}^{E} W\right] - \frac{1}{p}\sum_{H}^{K} W. \quad \ldots\ldots \quad (4)$$

The terms in this equation have interpretations like those in equation (2), except that the last term is the tangent of the angle which the slope line in the panel $H K$ makes with the horizontal. For a maximum value of S, the last member of (4) must change from positive to negative, which change will evidently occur just as some wheel enters the panel at K.

Equation (4) is somewhat more simple than equation (2) because the reaction polygon in this case is simply the weight line, *i. e.*, the reaction polygon for zero span. It will be convenient to go on and determine the segments of $A B$, in which the maximum shear is dominated by the successive wheels near the head of the train, as may be readily done by the method of Fig. 4, provided the points u_b u_d u_g be taken upon the zero reaction polygon, *i. e.*, upon the weight line, and provided the slope line of the panel having the inclination $W_1 \div p$ be taken instead of k_1 k_2, and the slope line of the panel having the inclination $(W_1 + W_2) \div p$ be taken instead of k_2 k_3, etc., etc. It is also to be noticed in this connection that the general expression for S which has been employed in obtaining equation (4) renders it evident that, other things being equal, the shear at C will be greater in case no part of the train stands between B and G, for b is negative for wheels at the right of B. Hence the train should be taken to consist of one locomotive and tender, or two locomotives and tenders followed by a uniform train or not, according as the one or the other supposition will cause the greater shear at C. The assumed uniform train should not in any case extend to the right of B.

In case there is no load at the right of B and the head of the train is near C, the construction for the greatest shear will be identical with that given in Part I, for the greatest shear in a single intersection truss of span $A B$.

Furthermore, it is evident from the general value of S just referred

to, that one maximum value of the shear occurs when the head of the train is near C, and another when it is near F, since f is negative for wheels standing to the right of F.

The concluding remarks of the last article respecting the comparison of the maximum shear at C in case the head of the train is near C, with the maximum shear at C in case the head of the train is near F, apply equally to the problem at present under discussion.

5. To discover what position of the train produces the greatest negative bending moment at any point C in the end span $B\,D$ of the cantilever $E\,D$. Figs. 1 and 6.

Let the sign of the bending moment as usual be taken as positive when produced by forces acting upward, then the moment M at C may be written:

$$M = -\,V_d\,d_c + \sum_{C}^{D} Wc$$

in which equation d_c denotes the distance from D to C; and c as before stated is the distance from C to any wheel, and is consequently negative for every wheel to the right of C.

But $V_d\,l_g = \sum_{D}^{G} Wg$, hence substituting for V_d we have

$$M = -\frac{d_c}{l_g}\sum_{D}^{G} Wg + \sum_{C}^{D} Wc$$

Now, let the train move an indefinitely small distance $\delta\,x$ toward the left and we find in the same manner as before:

$$\frac{\delta\,M}{\delta\,x} = -\left[\frac{d_c}{l_g}\,L_g - D_c\right] \qquad (5)$$

in which equation D_c denotes the load standing upon the segment $D\,C$. In order that M may reach a maximum negative value the parenthesis in (5) must change from positive to negative, *i. e.*, the last member of (5) must change from negative to positive. This change can occur in one way only, viz., some wheel may enter upon $C\,D$ at D, which will effect the change, for this will decrease L_g and increase D_c by an amount equal to the weight of the wheel which enters at D. And this is the only way that such a change can possibly be effected, for it is evident that when a wheel enters $D\,G$ at G, the effect is simply to increase L_g, and when a wheel leaves $C\,D$ at C the effect is to decrease D_c, but either of these changes tend to make the parenthesis of equation (5) positive and not negative. The graphical method of discovering which wheel must stand at D in order that M may reach a negative maximum at C is evidently identical with that employed in Part I, for discovering the position of the train which produces a maxi-

mum at D in a simple girder of span $C\,G$. The truth of this statement may be made evident as follows:

Without actually drawing any figure, but using the notation previously employed, let h_c and h_g be the points of the weight line at the verticals of C and G, respectively. Then equation (5) requires the two slope lines $h_c\,h_d$ and $h_d\,h_g$ to both lie on one side of $h_c\,h_g$, and the lines $h_c\,k_d$ and $k_d\,h_g$ to lie on the other side of $h_c\,h_g$. The construction may also be arranged thus: Prolong the slope lines $h_g\,h_d$ and $h_g\,k_d$ until they cut the vertical at C, then must h_c lie between these lines so prolonged if M is to reach a negative maximum.

Now, there may be several different positions of the train, which make the negative moment at C a maximum.

To find which bending moment is numerically greatest it will be sufficient to be able to effect a comparison of the bending moments at C for any two different positions of the cantilever with respect to the train.

Denote the respective positions of $C\,D\,G$, distant from each other by x, as before by $C'\,D'\,G'$ and $C''\,D''\,G''$, and using the value of M, already given in this article, we find:

$$M'' - M' = -\frac{d_c}{l_g}\left[\sum_{D'}^{G'} + \sum_{G'}^{G''} - \sum_{D'}^{D''}\right] W g'' + \frac{d_c}{l_g}\sum_{D'}^{G'} W(g'' - x)$$

$$+\left[\sum_{C'}^{D''} + \sum_{D'}^{D''} - \sum_{C'}^{C''}\right] W c'' - \sum_{C'}^{D'} W(c'' - x),$$

in which equation we have put $g' = g'' - x$, and $c' = c'' - x$.

Cancelling and then substituting $g'' = l + d''$ in the last term of the first bracket, and $c'' = d'' - d_c$ in the second term of the second bracket, and then cancelling again, we have finally

$$\frac{M'' - M'}{x\,d_c} = -\frac{1}{l_g}\left[\sum_{D'}^{G'} W + \sum_{G'}^{G''} W\,\frac{g''}{x} - \sum_{D'}^{D''} W\,\frac{d''}{x}\right]$$

$$+\frac{1}{d_c}\left[\sum_{C'}^{D'} W + \sum_{D'}^{D''} W\,\frac{d''}{x} - \sum_{C'}^{C''} W\,\frac{c''}{x}\right] \ldots\ldots\ldots (6)$$

It will be unnecessary to draw a figure here to give the graphical significance of equation (6), but using the notation before employed, let $u_c\,u_d\,u_g$ be points on the reaction polygon for the span x, at the verticals through $C'\,D'\,G'$ respectively. Equation (6) shows that in case the slope of the line $u_d\,u_g$ is steeper than that of $u_c\,u_d$, then is $M'' < M'$, and *vice versa*. But when $u_d\,u_g$ is the steeper, then is u_d below the slope line $u_c\,u_g$, and so $M'' < M'$.

Hence the construction is the same as that employed in Part I, for the case of the greater moment at D in a simple girder $C\ G$, except that in the present case the sign is reversed, *i. e.*, the smaller moment has the greater negative value, so that the moment which has actually the greatest numerical value is determined precisely as in the simple girder given in Part I.

The method which has now been given for discovering the position for the numerically greatest bending moment in the cantilever $E\ D$ suffices to find the position for the greatest stress in the flanges of a girder, or in the bars of the chord of a single intersection truss opposite the loaded pin points, or in the bars of both chords of a single intersection truss in the common case of intermediate suspenders or supports.

6. To discover the position of the train which causes the greatest stress in a bar $H\ K$, in the end span $B\ D$, of a single intersection cantilever truss $E\ D$, when the bar $H\ K$ is a member of the chord opposite C, a pin point of the unloaded chord.

In order to obtain the bending moment M, sustained by the truss itself, it will be necessary to subtract from the bending moment obtained in Art. 5, the bending moment in the panel $H\ K$. Since that moment was negative the result so obtained will be numerically the sum of these moments, and may be written :

$$M = -\frac{d_c}{l_g}\sum_{D}^{G} Wg + \sum_{C}^{D} Wc - \frac{c_h}{p}\sum_{H}^{K} Wk + \sum_{H}^{C} Wc,$$

in which equation p denotes the length of the panel $H\ K$, and c_h denotes the distance which H is to the left of C. Let the train move a small distance $\delta\ x$ toward the left, and we then obtain, by an algebraic process like that already employed, the expression :

$$\frac{\delta\ M}{\delta\ x} = -\left[\frac{d_c}{l_g}\sum_{D}^{G} W - \sum_{H}^{D} W + \frac{c_h}{p}\sum_{H}^{K} W\right] \ldots\ldots\ldots (7)$$

The meaning of this equation can be readily made evident by a figure similar to those already given. A rough diagram, such as the reader will at once supply from the following description, will make the interpretation clear. Let H be the pin point at the left end and K that at the right end of the panel $H\ K$ in the end span $B\ D$.

Call the points where the verticals at H, K, D and G cut the weight line h_h, h_k, h_d and h_g respectively. Draw the slope line $h_h\ h_k$ for the panel and call the point where the vertical at C cuts it j_c. Draw also the slope line $h_d\ h_g$ and prolong it until it intersects the vertical at C at the point t_c.

Now the last term in equation (7) expresses the height of j_c above h_h; the middle term is the height of h_d above h_h, and the difference of the two thus found is the difference in height between j_c and h_d. But the first term is the difference in height between t_c and h_d, and the entire quantity within the parenthesis in equation (7) is therefore merely an expression for the height $j_c\, t_c$, *i. e.*,

$$\frac{\delta\ M}{\delta\ x} = -\ j_c\ t_c,$$

in which equation $j_c\ t_c$ is positive in case t_c is above j_c and negative in case it is below.

But without drawing $h_d\ h_g$ or locating the point t_c at all, it is evident that $j_c\ t_c$ is positive or negative according as h_d lies above or below the slope line $j_c\ h_g$, as shown by a stretched thread.

In order that M may reach a negative maximum the quantity in the parenthesis in equation (7) must change from positive to negative, which change can occur at the instant when some wheel, say $W_d = h_d\ k_d$ reaches D, such that the slope line $j_c\ h_g$ cuts the weight line between h_d and k_d; but this change cannot occur when any wheel comes on at G, for that only increases the quantity in the parenthesis; nor can it occur when any wheel leaves the panel at H, for that has more effect to diminish numerically the negative term of the parenthesis than to diminish the positive term. But these are the only changes taken account of in equation (7), consequently the negative maximum, *i. e.*, the maximum numerical value of M, is reached only when some wheel reaches D.

In order to compare any two bending moments, negative maxima or not, at points of the train distant x from each other, denote the left hand position of the points $H\ C\ K$, etc., in the span $B\ D$ by $H'\ C'\ K'$, etc., and the right hand position of $H''\ C''\ K''$, etc. Then if we compare the expression for M in the first part of this article with that in the previous article, it appears that they differ in this, that the value of M we are now treating has in it a pair of terms due to the panel $H\ K$ and point C, and of the same signs as those due to the span $D\ G$ and point C. If we then compare the moments at C' and C'' in the panel $H\ K$, in a manner like that by which we obtained equation (6), we shall obtain an equation having in it all the terms found in equation (6), together with additional terms for the panel $H\ K$, which can be obtained as they were in the corresponding investigation in Part I, but which we can readily write by symmetry. We may therefore venture to write out the equation without further preliminaries, as follows:

$$\frac{M'' - M'}{x} = -\frac{d_c}{l_g}\left[\sum_{D'}^{G'} W + \sum_{G'}^{G''} W\frac{g''}{x} - \sum_{D'}^{D''} W\frac{d''}{x}\right] + \left[\sum_{C'}^{D'} W + \sum_{D'}^{D''} W\frac{d''}{x} - \sum_{C'}^{C''} W\frac{c''}{x}\right] - \frac{c_h}{p}\left[\sum_{H'}^{K'} W + \sum_{K'}^{K''} W\frac{k''}{x} - \sum_{H'}^{H''} W\frac{h''}{x}\right] + \left[\sum_{H'}^{C'} W + \sum_{C'}^{C''} W\frac{c''}{x} - \sum_{H'}^{H''} W\frac{h''}{x}\right] \quad \ldots\ldots (8)$$

Using the notation previously employed, the reader will be able by help of the description which follows to supply the figure needed in the interpretion of equation (8). Let u_c, u_d, u_g, etc., be the points on the reaction polygon x at the verticals through C', D', G', etc. Then is the quantity in the first parenthesis the height of u_g above u_d. Let the slope line $u_d\ u_g$ cut the vertical at C' in t_c. The first group term is the difference in height between t_c and u_d, and the first two terms taken together are equal to $u_c\ t_c$, *i. e.*, the height of t_c above the reaction polygon at the vertical through C', positive when t_c is above u_c, and negative when below.

The quantity in the third parenthesis is the height of u_k above u_h; and if the slope line $u_h\ u_k$ be drawn and we mark the point i_c where it cuts the vertical through C, then is the third group term the expression for the difference in height between u_h and i_c; and the fourth parenthesis is the height of u_c above u_h. Hence the last two terms together denote the height $u_c\ i_c$, negative in case i_c is above u_c and positive in case it is below. Hence, taken all together, the last member is the difference in height between i_c and t_c, *i. e.*,

$$\frac{M'' - M'}{x} = i_c\ t_c,$$

in which $i_c\ t_c$ is positive in case t_c is above i_c but negative in case it is below i_c.

Now, both M'' and M' are intrinsically negative, hence M'' is numerically greater than M' in case $i_c\ t_c$ is negative, *i. e.*, in case t_c falls below i_c.

It will be more convenient to stretch a thread so as to form the slope line $i_c\ u_g$, and not draw $u_g\ t_c$ at all; then in case u_d falls below $i_c\ u_g$ we have also t_c below i_c and *vice versa*. Hence, in case u_d falls below $i_c\ u_g$, M'' is numerically greater than M'; but when it is above, M' is numerically the greater.

It will be noticed that the bending moment and shear at a point C in the end span $B\ D$ depend in no way upon any loading except that standing between C and G in case of a girder, and upon that between H and K in addition, in case of a single intersection truss. The constructions

which have been given will, therefore, answer equally well for any point C in the end span $E\,A$, in which the greatest moment and shear will depend only upon the loading between C and F, or C and F, and H and K when $H\,C\,K$ lie in $E\,A$, while the load is still supposed to come on from G.

If the end spans $E\,A$ and $B\,D$ or $D\,G$ and $F\,E$ are not equal in length, it will be necessary first to put one end of the cantilever toward G, and then the other, which is equivalent to supposing the train first to come on at G and then at F.

It will also be best to compare numerically the greatest value of the moment in each end span when the train comes on at G with that when it comes on at F, in order to select the greater.

7. To discover the positions of the train which produce respectively the greatest and least bending moments, positive or negative, at any point C in the middle span $A\,B$ of the cantilever $E\,D$. By taking moments about C of the applied forces at the left C, the value of the bending moment M is found to be $M = V_a\,c_a - V_e\,(c_a + l_a) - \sum_{E}^{C} W c$.

Now, substituting the values of V_a and V_e, given in Art. 3, we have,

$$M = \frac{c_a}{l_b}\left[\sum_{E}^{D} W b - \frac{l_d}{l_g}\sum_{D}^{G} W g - \frac{l_a}{l_e}\sum_{F}^{E} W f\right] + \frac{l_a}{l_e}\sum_{F}^{E} W f - \sum_{E}^{C} W c.$$

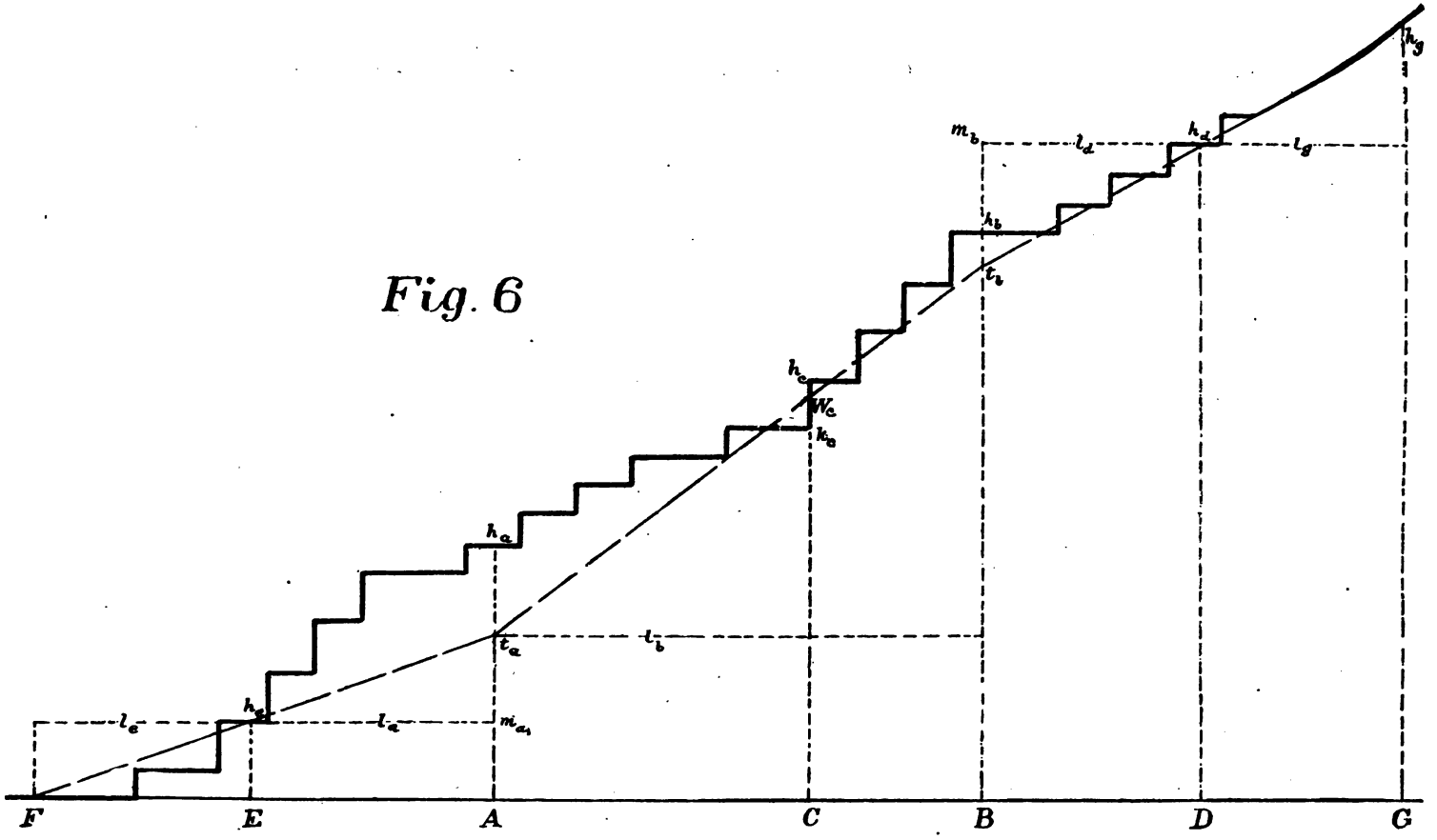
Now, let the train move an indefinitely small distance $\delta\,x$ toward the left, and we obtain by a method similar to that before employed:

$$\frac{\delta\,M}{\delta\,x} = \frac{c_a}{l_b}\left[\sum_{E}^{D} W - \frac{l_d}{l_g}\sum_{D}^{G} W - \frac{l_a}{l_c}\sum_{F}^{E} W\right] - \left[\sum_{E}^{C} W - \frac{l_a}{l_e}\sum_{F}^{E} W\right] \quad (9)$$

The second term within the first parenthesis of equation (9), is expressed in Fig. 6 by $m_a\,t_a$, and the third term by $m_b\,t_b$ as appears from similarity of triangles, hence the entire quantity within the parenthesis expresses the height of t_b above t_a. Similarly the last parenthesis of (9) expresses the height above t_a of k_c or of h_c according as $W_c\,(= h_c\,k_c)$ stands at the right or left of C. In case the slope line $t_a\,t_b$ cuts W_c between h_c and k_c, the value of M reaches a maximum at C, for it causes the last member of (9) to change from positive to negative. It will then be necessary to discover by trial which wheel weights on reaching C are cut by $t_a\,t_b$, and this will not be a thing difficult to do in a manner similar to the procedure in previous cases.

The value of M may also reach a maximum either when some wheel enters $D\,G$ at G, or when some wheel leaves $F\,E$ at F. For when a wheel enters at G that suddenly increases the slope of $h_d\,h_g$ and so decreases that of $t_a\,t_b$ without changing any other slope lines; and when a wheel leaves $F\,E$ at F, that decreases the slope of $h_f\,h_e$ and so increases the slope of both $t_a\,t_b$ and $t_a\,h_c$, while no other slope line is altered, but

Fig. 6



the slope of $t_a\ h_c$ will be increased much more than $t_a\ t_b$. Hence the last member of equation (9) may change from positive to negative, and so make M a maximum either just as a wheel comes on at G or goes off at F. The same result follows from considering the change in the numerical value of the terms which constitute the last member, equation (9). Again, in order to find a position of the train which will cause M to reach a minimum at C, it is necessary to find when the last member of (9) changes sign from negative to positive. This change can occur either when a wheel leaves $D\ G$ and enters upon $E\ D$ at D, or when it leaves $E\ D$ and enters upon $F\ E$ at E; for these two changes are exactly the reverse of those two just discussed, and they both of them evidently tend to make the last member of equation (9) positive. The same results appear from a consideration of the changes in the numerical value of the terms constituting the last member of equation (9).

In case the cantilever is a single intersection truss, and not a solid girder just as in the case previously treated, those positions of the train which cause M, the bending moment borne by the truss, to reach a maximum or a minimum value at a pin point C of the unloaded chord, *i. e.*, cause the maximum or minimum stress in a bar $H\ K$ of the loaded chord, are to be found by using the point j_c at the intersection of the slope line $h_h\ h_k$ with the vertical at C instead of the point h_c in the weight line. It will be unnecessary here to draw the figure, which will be readily supplied by the reader who examines Fig. 10, Part I, and equation (7) Part I; from which it appears that we can obtain the equation needed to treat this case by simply annexing to equation (9) Part II the terms relating to the panel $H\ K$, viz.:

$$-\left[\frac{c_h}{p}\sum_{H}^{K} W-\sum_{H}^{C} W\right]$$

It then appears just as shown in Part I, that the point j_c may be suddenly lifted upwards either when a wheel enters the panel $H\ K$ at K, or when a wheel leaves it at H in such a manner as to pass the point i_c situated upon $t_a\ t_b$. Hence the stress in $H\ K$ may reach a maximum value when some wheel stands at either of the four points H, K, F or G, and a minimum at either of the two points E or D. Which wheel of the train produces a maximum or a minimum at C must be discovered by trial by methods of procedure like those already employed.

It is to be noticed, however, in this connection, that in order to obtain the most unfavorable case on the whole, it will be necessary to consider the general value of M which precedes equation (9). From that equation it is evident that in order for M to be as great as possible, there should be no loading upon any part of $F\ G$ except upon $A\ B$ alone, and the train in the most unfavorable case should approximate as near this as the actual construction of locomotives and tenders will practically permit. In case it is possible to have no load except that

upon $A\ B$, the maximum moment due to the moving load is the same as that in the simple span $A\ B$.

Again, in order that the value of M may be the least possible at any point of $A\ B$ it appears from the same equation that there should be no loading upon $A\ B$. The most unfavorable case possible will then be assumed by supposing the rest of $F\ G$ to be covered by locomotives with tenders and trains.

In treating this last combination of two assumed trains it will be best to find the train position for a minimum at C due to a train approaching from G, when its first wheel is near B, and then a train position for a minimum at C due to a train approaching from F when its first wheel is near A. This combination will usually produce the most unfavorable case of a minimum or negative maximum at C.

8. In order to discover which of two positions of the train causes the greater bending moment at any point C of the span $A\ B$, the following comparison is given of the two values of M at any distance x from each other along the train.

The steps of the analysis are strictly analogous to those before employed, and need not be given in quite such detail as in the previous demonstrations. Employing the value of M, from which equation (9) is derived, we have:

$$\begin{aligned}
M''-M' = {} & \frac{c_a}{l_b}\left[\sum_{E'}^{D'} + \sum_{D'}^{D''} - \sum_{E'}^{E''}\right] W b'' - \frac{c_a}{l_b}\sum_{E'}^{D'} W (b''-x) \\
& - \frac{c_a\, l_d}{l_b\, l_g}\left[\sum_{D'}^{G'} + \sum_{G'}^{G''} - \sum_{D'}^{D''}\right] W g'' + \frac{c_a\, l_d}{l_b\, l_g}\sum_{D'}^{G'} W (g''-x) \\
& - \frac{c_b\, l_a}{l_b\, l_e}\left[\sum_{F'}^{E'} + \sum_{E'}^{E''} - \sum_{F'}^{F''}\right] W f'' + \frac{c_b\, l_a}{l_b\, l_e}\sum_{F'}^{E'} W (f''-x) \\
& - \left[\sum_{E'}^{C'} + \sum_{C'}^{C''} - \sum_{E'}^{E''}\right] W c'' + \sum_{E'}^{C'} W (c''-x)
\end{aligned}$$

in which equation $c_b = c_a - l_b$. Hence

$$\begin{aligned}
\frac{M''-M'}{x} = {} & \frac{c_a}{l_b}\left[\sum_{E'}^{D'} W + \sum_{D'}^{D''} W\ \frac{(d''-l_a)}{x} - \sum_{E'}^{E''} W\ \frac{(l_a+l_b+e'')}{x}\right] \\
& - \frac{c_a\, l_d}{l_b\, l_g}\left[\sum_{D'}^{G'} W + \sum_{G'}^{G''} W g'' - \sum_{D'}^{D''} W\ \frac{(d''+l_g)}{x}\right] \\
& - \frac{c_b\, l_a}{l_b\, l_e}\left[\sum_{F'}^{E'} W + \sum_{E'}^{E''} W\ \frac{(e''-l_e)}{x} - \sum_{F'}^{F''} W\ \frac{f''}{x}\right] \\
& - \left[\sum_{E'}^{C'} W + \sum_{C'}^{C''} W\ \frac{c''}{x} - \sum_{E'}^{E''} W\ \frac{(c_a+l_a+e'')}{x}\right]
\end{aligned}$$

$$\therefore \frac{M''-M'}{x} = \frac{c_a}{l_b}\left[\sum_{E'}^{D'} W + \sum_{D'}^{D''} W\frac{a''}{x} - \sum_{E'}^{E''} W\frac{e''}{x} - \frac{l_d}{l_g}\left(\sum_{D'}^{G'} W + \sum_{G'}^{G''} W\frac{g''}{x} - \sum_{D'}^{D''} W\frac{d''}{x}\right) - \frac{l_a}{l_e}\left(\sum_{F'}^{E'} W + \sum_{E'}^{E''} W\frac{e''}{x} - \sum_{F'}^{F''} W\frac{f''}{x}\right)\right] - \left[\sum_{h''}^{C'} W + \sum_{C'}^{C''} W\frac{c''}{x} - \sum_{E'}^{E''} W\frac{e''}{x} - \frac{l_a}{l_e}\left(\sum_{F'}^{E'} W + \sum_{E'}^{E''} W\frac{e''}{x} - \sum_{F'}^{F''} W\frac{f''}{x}\right)\right] \quad \ldots (10)$$

It will be unnecessary to do more than to describe the figure which expresses the relations in the last member of (10), as it can be easily supplied by the reader. Let $u_f\, u_e$, etc., be the reaction polygon for the span x. Now prolong the slope line $u_f\, u_e$ until it intersects the vertical through A at t_a. Likewise prolong $u_d\, u_g$ until it intersects the vertical through B at t_b. Let i_c be the point where the line $t_a\, t_b$ intersects the vertical through C. Then is

$$\frac{M''-M'}{x} = i_c\, u_c$$

in which u_c is the point of the reaction polygon at the vertical C. In case u_c is above i_c then $M''<M'$, and *vice versa*.

In case it is desired to compare the stresses in a bar HK of the loaded chord of single intersection truss cantilever we must merely employ the point j_c at the intersection of $u_h\, u_k$ and the vertical at C instead of the point u_c, which statement appears sufficiently evident from previous similar discussions.

9. The double intersection truss with end supports.

It is usual to assume that a truss of this kind consists of two components, each of which is a single intersection truss, and that these two components are superposed upon each other in such a manner that while they have their chord members in common, the web members of the two systems are more or less distinct from each other. The mutual independence of the two systems of web members is usually such that it is regarded as sufficiently accurate for practical purposes to assume that any load applied at a panel point of one system causes no stresses in the web members of the other system. This assumption is taken as the basis of the following investigation which has for its object the establishment of graphical or semi-graphical methods, in connection with the reaction polygons due to the wheel weights of the train, for discovering the train

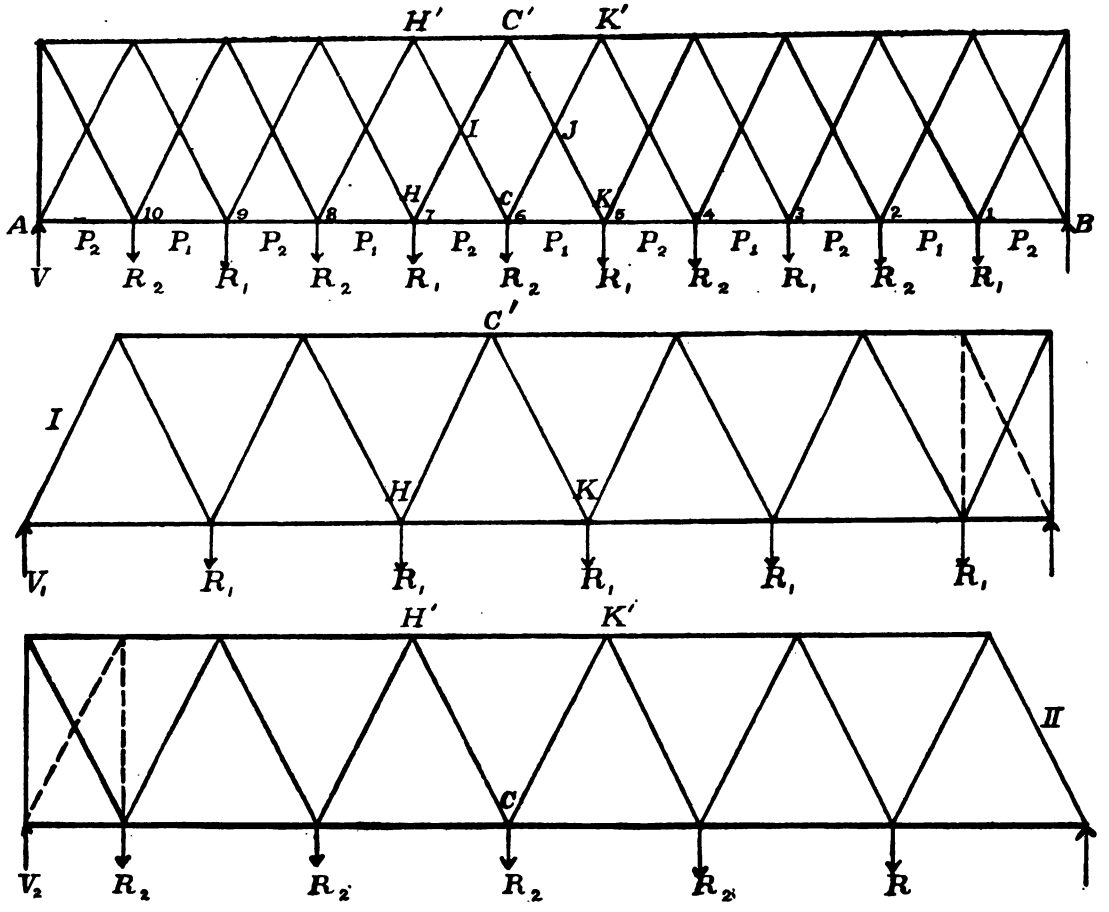


Fig. 7

position which causes the greatest stress in any member of a given double intersection truss.

10. To discover the train position which produces a maximum stress in a given web member of a double intersection truss having inclined web members as shown in Fig. 7.

This maximum stress occurs, as is well known, coincidently with the greatest shear in that panel of the component truss in which the web member is situated, for the shear is the vertical component of the stress in the member.

In order then to investigate the algebraic condition for a maximum value of the shear in a panel of a component truss, let Fig. 7 represent a double intersection truss and its components I and II; and let it be required to find the condition for a maximum shear in the panel HK of component I, *i. e.*, in the two web members of this panel.

Denote the pin load at any panel point of component I by R_1 and that at any panel point of II by R_2. Also designate by P_1 the load in any panel just at the left of any one of the panel points of I, *e. g.*, just at the left of H or K; and designate by P_2 the load in any panel just at the left of either of the panel points of II, as shown in Fig. 7.

Then any R_1, as for example R_1 at K, is due to P_1 at its left and P_2 at its right; while any R_2, as for example R_2 at C is due to P_2 at its left and P_1 at its right.

Then

$$R_1 = \frac{1}{p}(P_2\, c_2 - P_1\, c_1)$$

Let the panel points in II, which are next left and next right of K in I, be denoted respectively by C_1 and C_2, and let P_1 and P_2 be the panel loads respectively at the left and right of R_1 at K, while c_1 and c_2 denote respectively the distances of the centers of gravity of P_1 and P_2 from C_1 and C_2, these distances being taken as positive toward the left and negative toward the right. We see that c_1 in Fig. 8 is negative.

Now the shear S_1 in any panel HK of I may be written—

$$S_1 = \sum_A^B R_1 \frac{b_1}{l} - \sum_A^C R_1$$

in which equation b_1 denotes the distance of any pin load R_1 from B. Substitute in this equation the value of R_1 just found, and we find—

$$p\, S_1 = \sum_A^B (P_2\, c_2 - P_1\, c_1)\, \frac{b_1}{l} - \sum_A^C (P_2\, c_2 - P_1\, C_1)$$

Now let the train move an infinitesimal distance δx toward the left. In this new position c_2 and c_1 become respectively $c_2 + \delta x$ and $c_1 + \delta x$, and we find after reduction—

$$p\frac{\delta S_1}{\delta x} = \sum_{A}^{B}(P_2 - P_1)\frac{b_1}{l} - \sum_{A}^{C}(P_2 - P_1)$$

In order to put this last equation into a form somewhat more symmetrical and convenient for discussion, multiply the last summation by $\frac{b_1 - a_1}{l}$ ($= 1$), separate the summation from A to B into two parts, one part from A to C, and the other from C to B, and then cancel. The result may then be put in the form—

$$\frac{l}{2}\frac{\delta S_1}{\delta x} = \frac{1}{2}\sum_{A}^{C}(P_2 - P_1)\frac{a_1}{p} + \frac{1}{2}\sum_{C}^{B}(P_2 - P_1)\frac{b_1}{p} \quad \ldots(11)$$

In which equation a_1 is the distance from A to any R_1 and is to be taken with the P_2 and P_1 respectively just at the right and left of this R_1, while b_1 is the similar distance from B; consequently $\frac{b_1}{p}$ is an integer as is $\frac{a_1}{p}$. Every a_1 in (11) is negative and every b_1 is positive.

For brevity write equation (11) thus:

$$\frac{l}{2}\frac{\delta S_1}{\delta x} = -\frac{1}{p}(A_1 - B_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots (11_1)$$

in which $-A_1$ is one-half the summation from A to C, and B_1 is one-half that from C to B.

Now, in order to see how S_1 may reach a maximum it is necessary to trace the variations in the value of the last member of (11) and discover under what circumstances it changes from positive to negative. It is so easy to get a clear idea of the numerical changes in the value of (11) in the simple case of a uniform train of unlimited length coming on at B, and this case enables us to gain such an insight into the usual cases of unequal loading, that we shall consider that special case first.

For brevity, designate the first pin point of the loaded chord in I at the left of B as joint 1; the next pin point of I as joint 2, etc.

Now as the head of a uniform train moves from B toward joint 1, that panel load of equation (11) which first comes into existence is the P_2 next at the right of joint 1. This furnishes a positive term in B_1, so that δS_1 is necessarily positive until the head of the train reaches joint 1. As the head of the train moves on toward the left of joint 1, the next panel weight P_1 comes into existence, and by the time this whole panel is covered by the train the quantity $P_2 - P_1$ for joint 1 vanishes and $\delta S_1 = 0$. Up to this point δS_1 has not at any instant been negative, and so the condition for a maximum value of S_1 has not been satisfied.

If now the train continues to move toward the left, it is evident that a series of statements can be shown to be true respecting joint 2

precisely like those already shown to hold respecting joint 1; and that meanwhile that part of B_1 due to joint 1 is zero and remains zero during all the subsequent motion.

When the two panels about joint 2 are both covered, then $P_2 - P_1$ becomes zero for this joint also, and remains so. This state of affairs continually repeats itself at the head of the uniform train until it reaches C, *i. e.*, the value of δS_1 has at no time been negative. Hence S_1 has in general been increasing as the head of the train progressed toward C, though the rate of increase has not been uniform. When now the head of the train arives at C, δS_1 vanishes, but as soon as it has passed beyond C towards A, then a term of the type P_2 in A_1 first comes into existence, and δS_1 then becomes negative for the first time.

The condition for a maximum value of S_1 is thus fulfilled when the head of the uniform train reaches C, and it is not fulfilled again, as δS_1 does not again become positive.

To return now to the more general case where a train of unequal wheel weights enter upon the span at B, we see that the first P_2 comes into existence as the head of the train approaches joint 1, and the first P_1 makes its appearance when the head of the train has passed joint 1. In case $P_2 - P_1$ for joint 1 becomes negative before there is any pin load on joint 2, the condition for a maximum value of S_1 is then fulfilled, but the maximum so reached is not in general so great as that reached when the head of the train is near C. For it is evident, as the train proceeds toward the left and the P_2 for joint 2 comes into existence, that we soon have $\delta S_1 > 0$ for two reasons; first, because the P_2 at joint 2 must in general exceed $P_2 - P_1$ at joint 1; and secondly, the b_1 of joint 2 is greater than the b_1 of joint 1.

It is likewise seen that as the head of the train progresses towards C the value of δS_1 can be negative only occasionally and for short intervals, so that the sign of δS_1 is controlled and kept positive most of the time, just as in case of a uniform train, by the relatively large value of $P_2 b_1$ in the panel near the head of the train. This insures that S_1 shall, with insignificant exceptions, be continually increasing as the train progresses until W_1, the head of the train, is in the neighborhood of C. Consequently the greatest value of S_1 occurs when W_1 is not far from C, a result in accordance with what is well known to be the case for the simple girder and for the single intersection truss.

When any wheel, as for example W_1, stands at C, it evidently tends to make the last member of (11) negative, for in case it is regarded as infinitesimally distant and toward the right of C, it is then part of the P_1 at K, and so enters the second summation of (11) as a negative quantity; but in case it is regarded as standing just at the left of C, it forms part of the P_2 at H, and consequently enters the first summation of (11) as positive, with a negative multiplier a_1. Whether any wheel, as W_1, has a greater influence to render (11) negative, and so produce a

maximum of S_1 by standing just at the right or at the left of C, depends upon the relative magnitude of its b_1 and its a_1 in these positions; *i. e.*, in case $A\,H < K\,B$, or what is the same thing, in case $A\,C < C\,B$, then any wheel has more influence to render (11) negative by standing at the right of C than at the left. But when $A\,C > C\,B$ the wheel has more influence standing at the left of C. Hence it follows that a maximum shear may be caused in any panel $H\,K$ in the right half of the span by the passage of W_1, or some other wheel over the panel point C of component II, intermediate between H and K, but the maximum shear cannot be so caused in any panel of the left half of the span.

Hence when there are counterbraces they can reach their greatest stress in this manner only, though they may not reach it in this manner, but instead, at the instant some wheel passes K or some other panel point of component I, between C and B, as will be made to appear later.

Since W_1 cannot be very distant from C when S_1 reaches its greatest value, suppose the train to occupy initially a position such that W_1 is just at the right of C.

We shall now proceed to find out whether the last member of (11) is positive or negative.

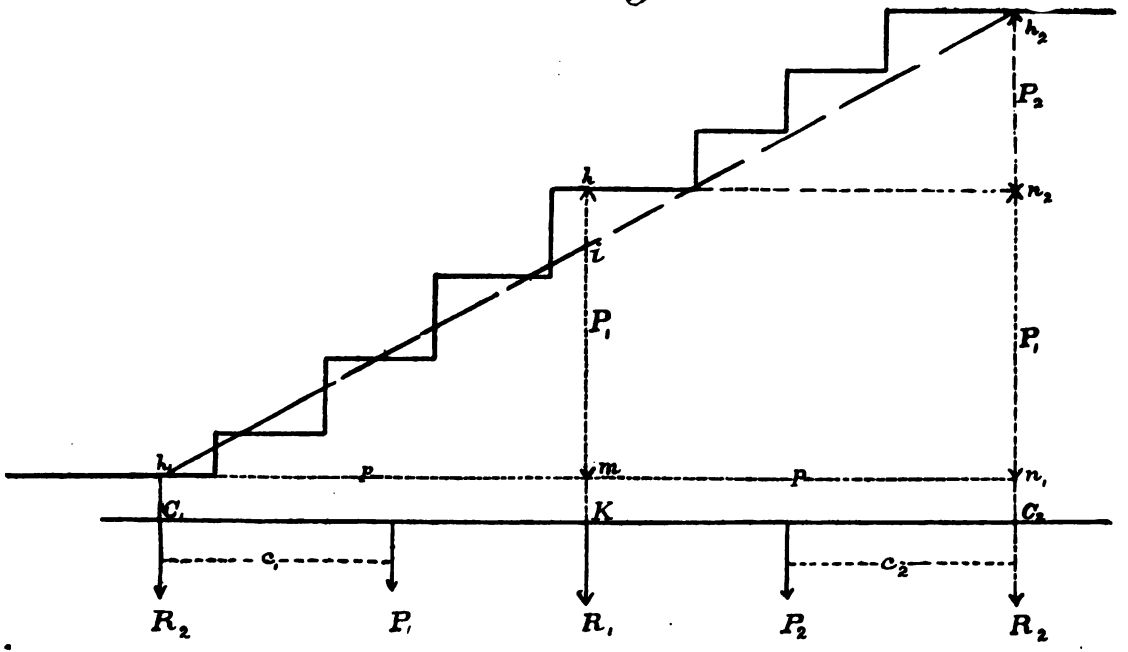
Equation (11) is of a form which admits of ready computation, especially since the value of $\frac{1}{2}(P_2 - P_1)$ can be obtained for each R_1 by inspection from the diagram of the weight line as shown in Fig. 8, thus: Mark the panel points of the truss upon the edge of a strip of paper and place it horizontally upon the diagram of the weight line in such a position that C will lie in the vertical at W_1. Let K in Fig. 8 be any panel point of component I, and let C_1 and C_2 be adjacent panel points of the same chord in component II.

Then is P_1 the height of h above h_1 and P_2 that of h_2 above h. Also $P_2 + P_1$ is the height of h_2 above h_1 and $\frac{1}{2}(P_2 + P_1)$ that of i above h; hence $h\,i = \frac{1}{2}(P_2 + P_1) - P_1 = \frac{1}{2}(P_2 - P_1)$, which quantity is positive in case i is above h, but negative in case i is below h. The value of $\frac{1}{2}(P_2 - P_1)$ at each panel point of component I can thus be read off by scale on the weight diagram without defacing it in any way. Take the numerical value of $\frac{1}{2}(P_2 - P_1)$ thus obtained at each panel point of I, multiply it by $b_1 \div p$, the integer expressing the number of panels the point is from B, and finally take the algebraic sum of these products from C to B.

It will usually be possible to discover whether the last member of (11) is positive by taking the terms at only one or two of the panel points near the head of the train. The result of this brief numerical computation of (11) will show whether $\delta\,S_1$ is positive or negative when W_1 is just at the right of C.

Let us discuss the two cases in which, first, (11) is positive, and, secondly, negative with W_1 just at the right of C. First, in case (11)

Fig. 8



is positive and C is in the right half of the span, (11) may become negative and make S_1 a maximum, and is likely to do so when W_1 passes C.

The only numerical changes in (11) which will in general occur as W_1 passes from a position just at the right to one just at the left of C, is to increase B_1 by the quantity $W_1 . K B$ and decrease $- A_1$ by $W_1 . A H$, or on the whole decrease (11) by

$$\frac{1}{2} W_1 (A H - K B) = \frac{1}{2} W_1 (A C - C B) = \frac{1}{2} W_1 (2 c_a - l).$$

If the subtraction of this quantity renders (11) negative, then W_1 at C makes S_1 a maximum and in all ordinary cases this is the greatest maximum. If, however, the subtraction of this quantity does not render (11) negative, the train must be moved still further toward the left. During this motion the first change that can occur having a tendency to make (11) negative will take place as soon as either one of the panel points K, etc., of component I in the segment $C B$, or the point C is next passed by any wheel. The next wheel that can pass C is W_2, so that one or other of the points K, etc., between C and B will in general be passed by some wheel before W_2 reaches C. It is to be noticed at the same time that the change in the value of (11) caused by any wheel in passing a panel point of component II between C and B does not tend to make (11) negative but positive.

If in moving the train from its position with W_1 just at the left of C until some wheel, as W_n, passes some point of component I between C and B, as K_n, no wheel has meanwhile passed any point of component II, then the value of (11) will be unchanged until W_n passes K_n.

Whether (11) is unchanged or not, it will be possible to obtain the quantity to be subtracted from (11) when W_n passes K_n as follows. At K_n the P_1 will be increased by W_n, and the P_2 decreased by W_n, consequently $\frac{1}{2} (P_2 - P_1)$ will, on the whole, be decreased by W_n and the quantity to be subtracted from (11) is $W_n \frac{b_1}{p}$, in which expression $b_1 = K_n B$, the distance from the point K_n, passed by the wheel, to B. If the decrease of (11) by the quantity $W_n \frac{b_1}{p}$ changes the sign of (11), then S_1 reaches a maximum, and in general its greatest maximum with W_n at K_n. If, however, (11) does not become negative by this decrease, the train must be still further moved toward the left until some wheel passes a joint of component I which changes the sign of (11), or until some wheel passing C makes (11) negative. When joint C is in the left half of the span the discussion differs from that just given for the right half only in this, that the possibility of a maximum when some wheel passes C is shut out, as was shown previously.

Secondly, in case (11) is negative when W_1 stands just at the right

of C, then (11) must have become negative before the train reached this position, which could have occurred only as some wheel W_n passed same joint K_n of component I very shortly before W_1 reached C. Move the train backward until it first occurs that a wheel passes a joint of component I. Call the joint K_n and the wheel weight W_n, and see whether the decrement $W_n \frac{b_1}{p}$ due to the passage of W_n across K_n from right to left has been the cause of the negative value of (11). If the wheel W_n is one near the head of the train, it is in general the one which has rendered (11) negative; but if the wheel is one passing a joint near B, it is not so likely to be the one which has rendered (11) negative. By continuing the examination, the wheel will soon be discovered which has rendered (11) negative. It is taken for granted in this discussion that the head of the train will not arrive at the joint of component II next left of H before the greatest value of S_1 has been reached. If in any exceptional case W_1 should arrive at the joint next to the left of H before S_1 reaches a maximum, it might then produce a maximum of S_1 by passing that joint, for it would increase P_2 and decrease P_1 at that joint and so tend to make (11) negative.

Furthermore, in considering the position of W_1 with respect to C, it will be noticed that in case the joint C is one situated near B, it is impossible for W_1 to pass C without making (11) negative, though it may do so in case C is a joint situated near A. Besides this, it appears that the value of the quantity $(P_2 - P_1)$ at a joint K near the head of the train, has a more decisive effect in controlling the sign of δS_1 when C is near B than when C is near A, and this alteration goes on from joint to joint as we proceed from B to A. The total effect of this is to allow the head of the train to stand a little further to the left with respect to C, the further C is toward the left. This agrees with the result already found to hold for the simple girder, and for the single intersection truss, in both of which the train advances with respect to the cross section, at which the maximum shear occurs as the cross section advances from B toward A. The advance here spoken of takes place in each case by jumps from wheel to wheel; but in the double intersection truss, as just shown, the law is more complicated than in the other two cases, as the maximum shear depends upon other wheels besides the first one or two at the head of the train. This prevents our finding the segment in which the greatest shear is dominated by any particular wheel, though it might be possible to determine this approximately. Such a determination seems, however, to be nearly useless in view of what has been just stated as to the continued advance of the train with respect to C. For when the train position has been found which makes the shear a maximum at joint 1, then the maximum shear at joint 2 is due either to a train position which is the same relatively to joint 2, or to a position slightly in advance of this; and so on for joint 3, etc. It is to be specially

noticed that the investigation which has been made of the train position for maximum shear in the web members of any panel has been entirely independent of any assumption as to pattern of the truss, except that it consists of two simple trusses superposed in such a manner that the equidistant joints belong alternately to each of the components. This has important consequences, viz.: the investigation not only applies to the form of truss shown in Fig. 7, but to any modifications of it, obtained by changing the inclination of any or all of the web members. For example, it applies to the case where the end posts are inclined, according to the usual design as shown in the dotted lines, Fig. 7. Again, the ordinary double intersection Whipple Truss with inclined end posts and vertical intermediate posts, is also included in our investigation, provided we may assume the vertical tie at the first panel point to belong exclusively to one component, as we are entitled to do without appreciable error, and we also make the like assumption respecting the vertical tie at the last panel point.

Likewise the Post Truss, or any other form of single intersection truss is also included in equation (11). The only point to be considered is whether the maximum shear, as thus determined, is to be sustained by principal web members or by counters; the latter is the case when the shear in one half of the span is of the same sign as that in the end post of the other half span.

11. The train position which produces a maximum stress in the web members of a multiple intersection truss may also be discovered by equation (11).

This is evident when we notice that (11) was derived by taking into account only those pin loads designated as R_1 which are supported by component I, and it is of no consequence how many other components there may be in the complete truss, as (11) is independent of all of them.

Hence in applying equation (11) to discover the position of the train for a maximum value of S_1 in a multiple intersection truss it will be necessary to take the value of $(P_2 - P_1)$ at each panel point only of the component whose web member we are considering, and effect the summations expressed in equation (11) for that component alone. In the multiple truss there will be several panel points of other components intermediate between successive panel points, as H and K, of the component under consideration; but in taking the summations in equation (11) it will make no difference which of the intermediate points be regarded as C.

Suppose, for example, that the intermediate panel point nearest H is denoted by C_1, and that nearest K by C_2, then the first summation of equation (11) may conveniently be taken from A to C_1 and the second from C_2 to B.

If the panel points of the truss be marked upon the edge of a strip of

paper the computation of equation (11) can be made as before, but with this difference, that while in a double intersection truss all the wheel weights upon the truss enter either one or other of the summations, in any other multiple intersection truss only those wheel weights are included in the summations which stand in the panels adjacent to the panel points of the component under consideration. The wheel weights upon other panels than these have no influence upon the value of S_1.

The circumstances under which the last member of (11) may change sign from positive to negative, and so cause a maximum value of S_1, are these: Some wheel, as W_n, may pass one of the panel points of component I between C_2 and B, and so decrease (11) by $W_n \frac{b_1}{p}$, or, some wheel W_n may enter upon or leave some panel included in the summation between A and C_1, and so decrease (11) by $W_n \frac{a_1}{p}$.

It will be noticed when any wheel enters upon or leaves any panel included in the summation between C_2 and B that it tends to make (11) positive; and when any wheel passes a panel point of I between A and C_1 it has a like tendency.

The greatest value of S_1 will be found when the head of the train is at or near C_1, and its precise position can be discovered by a method like that previously explained as applicable to the double intersection truss. It will only be necessary to designate each component as it successively comes under consideration as component I.

Remarks like those made at the end of the previous article are also true for the multiple intersection truss, viz.: the pattern of the truss is of no account, provided we may assume with sufficient correctness that the pin loads of one component are not transmitted to the piers through the web members of the other components.

12. To discover the train position which produces a maximum shearing stress in a web member of a double intersection truss with intermediate supports $O\,I$, $O\,J$, etc., Fig. 9.

The double intersection truss with intermediate supports, as shown in Fig. 9, is a design so commonly employed and so economical of material, that we proceed to investigate the train positions for maximum stresses in the web members by a method similar to that just employed for the double intersection truss.

As this is a modification of the double intersection truss previously treated, let the notation previously employed be retained with certain slight modifications and additions.

Designate the loads at the pin points H, K, etc., of component I by R_1', those at C, etc., of II by R_2' and those at the intermediate pin points next at the left of each R_1'' (as at Q, etc.) by R_0', but those next left of each R_2' (as at O, etc.) by R_0''. Furthermore, let the panel load P_2 upon $H\,C$ consist of two parts, P_2'' upon $H\,O$ and P_2' upon $O\,C$;

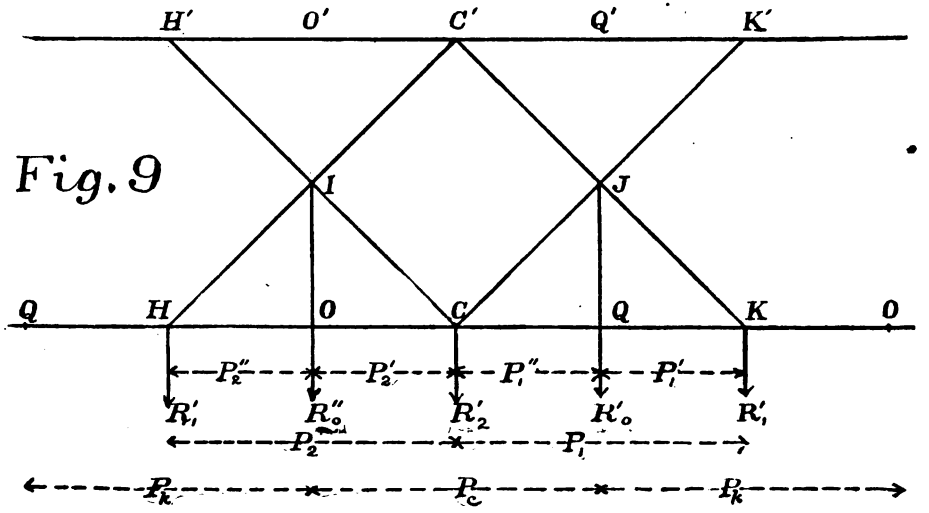


Fig. 9

and similarly for P_1 upon $C\,K$, so that $P_2 = P_2'' + P_2'$ and $P_1 = P_1'' + P_1'$. Moreover, let the load upon $O\,Q$ be $P_c = P_2' + P_1''$, then is P_c the load in the two half panels about any point C of II, and similarly let $P_k = P_1' + P_2''$ be the load in the two half panels about any panel point H or K of I.

Then, by our previous notation for distances, taken as positive toward the left, and by taking $H\,C = p = C\,K$ and $O\,C = \frac{1}{2}\,p = C\,O$, etc., we have the equations,

$$\begin{aligned}
\tfrac{1}{2}\,p\,R_0'' &= P_2'\,c - P_2''\,h \\
\tfrac{1}{2}\,p\,R_0' &= P_1'\,k - P_1''\,c \\
\tfrac{1}{2}\,p\,R_1' &= P_2''\,o - P_1\,q \\
\tfrac{1}{2}\,p\,R_2' &= P_1''\,q - P_2'\,o
\end{aligned}$$

1° The shear in the web members of component I at any point between O and Q may evidently be written in the form

$$S_1 = V_1 - \sum_A^C [R_1' + \tfrac{1}{2}\,(R_0' + R_0'')],$$

for R_1' is entirely supported by I, and by reason of the symmetry of the web members at I and J, each intermediate pin load R_0' and R_0'' is supported half by component I and half by II.

By taking moments about B we find the load upon component I causes a reaction at A to the amount

$$V_1 = \frac{1}{l}\sum_A^B [(R_1'\,b_1 + \tfrac{1}{2}\,R_0'\,(b_1 + \tfrac{1}{2}\,p) + \tfrac{1}{2}\,R_0''\,(b_1 - \tfrac{1}{2}\,p)].$$

Substitute this value of V_1 in the previous equation and we have

$$2\,S_1 = \frac{1}{l}\sum_A^B [(2\,R_1' + R_0' + R_0'')\,b_1 + \tfrac{1}{2}\,(R_0' - R_0'')\,p] - \sum_A^C [2\,R_1' + R_0' + R_0''].$$

In this equation introduce the values of the various pin-loads R_1', R_0', R_0'', previously given, and then let the train move a small distance $\delta\,x$ toward the left.

We then finally obtain the equation :

$$p\,\frac{\delta\,S_1}{\delta x} = \frac{1}{l}\sum_A^B [(P_2 - P_1)\,b_1 + \tfrac{1}{2}\,(P_k - P_c)\,p] - \sum_A^C (P_2 - P_1),$$

or,

$$p\,\frac{\delta\,S_1}{\delta\,x} = \sum_A^C (P_2 - P_1)\,\frac{a_1}{l} - \sum_C^B (P_2 - P_1)\,\frac{b_1}{l} + \frac{p}{2\,l}\,(P_k - P_c) \ldots (12)$$

By comparing this equation with equation (11), it appears that it differs from (11) only by the addition of a final term which will, under all ordinary circumstances, be small. The presence of this term in (12) makes it possible, however, that the maximum value of S_1 may occur as some wheel passes any one of the points Q, but not as it passes any of the points O. S_1 may also be a maximum in (12) under all the same circumstances as S_1 in equation (11), which need not be here repeated.

2°. To find the value of S_1 between Q and K, *i. e.*, in JK, it is necessary to subtract the quantity $\frac{1}{2} R_0'$ at Q from the value of S_1 already found between O and Q.

Hence we must subtract from (12) the quantity

$$\tfrac{1}{2} p \frac{\delta R_0'}{\delta x} = \sum_{C}^{K} (P_1' - P_1'')$$

This term of correction, whose value may be large, when appended to equation (12) makes it evident that the maximum value of S_1 in JK may occur just as some wheel passes K.

3°. Similarly to find the value of S_1 between H and O, *i. e.*, in HI, it is evidently necessary to add the quantity $\frac{1}{2} R_0''$ at O to the value of S_1 already found between O and Q. Hence we must add to (12) the correction

$$\tfrac{1}{2} p \frac{\delta R_0''}{\delta x} = \sum_{H}^{C} (P_2' - P_2'')$$

which makes it evident that the maximum value of S_1 in HI may occur just as some wheel passes O.

13. To discover the train position which produces the greatest stress in any member of the loaded chord of a double intersection truss with parallel chords, as in the bar CK, Fig. 7.

Let the notation be like that employed in connection with Fig. 7, viz.:

R_1 = the load at any pin point of I,
R_2 = the load at any pin point of II,
l = length of span ; h = depth of truss,
c' = the distance of any R_1 from C',
k' = the distance of any R_2 from K',

these distances being both taken as positive toward the left,

$$c_a' = C'A, \quad k_a' = c_a' + p = K'A.$$

T_1 = tension in chord member of I opposite the joint C'.
T_2 = tension in chord member of II opposite the joint K'.
$T = T_1 + T_2$ = total tension in CK.

Take moments about C' of all the forces acting on component I at the left of C', and we have

$$h T_1 = V_1 c_a' - \sum_{A}^{C'} R_1 c'$$

Similarly take moments about K' of all forces acting on component II, and we have

$$h\,T_2 = V_2\,k_a' - \sum_A^{K'} R_2\,k'$$

in which equations the reactions at A of components I and II are respectively

$$V_1 = \frac{1}{l}\sum_A^B R_1\,b_1,\quad V_2 = \frac{1}{l}\sum_A^B R_2\,b_2,$$

b_1 and b_2 being distances of R_1 and R_2 respectively from B. Add the two equations of moments just given, replace $T_1 + T_2$, V_1 and V_2 by their values already given, and finally substitute for R_1 and R_2 their values from the equations

$$p\,R_1 = P_2\,c_2 - P_1\,c_2,\quad p\,R_2 = P_1\,k_1 - P_2\,k_2.$$

The value of R_1 has already been obtained in connection with Fig. 8. The value of R_2 is derived in a similar manner, as will be evident when we notice in Fig. 8 that the panel loads about K in component I follow each other from left to right in the order $P_1\,P_2$, so that the value of R_1 is obtained from consideration of three consecutive panel points in the order $C_1\,K\,C_2$, while the panel loads about any panel point, C in component II follow in the reverse order $P_2\,P_1$; hence the value of R_2 is obtained from the consideration of three consecutive panel points in the order $H\,C\,K$; or, taking a more systematic notation, we may use instead the letters $K_2\,C\,K_1$. The equation finally obtained may be written in the form

$$p\,h\,T = \frac{c_a'}{l}\sum_A^B (P_2\,c_2 - P_1\,c_1)\,b_1 - \frac{k_a'}{l}\sum_A^B (P_2\,k_2 - P_1\,k_1)\,b_2$$
$$- \sum_A^{C'} (P_2\,c_2 - P_1\,c_1)\,c' + \sum_A^{K'} (P_2\,k_2 - P_1\,k_1)\,k'.$$

Now let the train move a small distance $\delta\,x$ toward the left, then will each of the distances c_1, c_2, k_2, k_2 be increased by $\delta\,x$.

Let T be changed by this movement to $T + \delta\,T$; we then obtain by subtraction and division,

$$p\,h\,\frac{\delta\,T}{\delta\,x} = \frac{c_a'}{l}\sum_A^B (P_2 - P_1)\,b_1 - \frac{k_a'}{l}\sum_A^B (P_2 - P_1)\,b_2 - \sum_A^{C'} (P_2 - P_1)c'$$
$$+ \sum_A^{K'} (P_2 - P_1)\,k'.$$

In order to trace changes in the sign of this equation more readily, let us make certain transformations.

Since b_1 refers to R_1 and not to R_2, it is evident that the b_1 of any P_1 extends from B to the right hand extremity of the panel in which this P_1 stands; but, since b_2 refers to R_2 only, the b_2 of any P_1 extends from B to the left extremity of the panel in which P_1 is, and *vice versa*, for the b_1 and b_2 of any P_2.

Hence	for	any	P_1	we	have	$b_2 - b_1 = p,$
but	"	"	P_2	"	"	$b_1 - b_2 = p.$
Similarly	"	"	P_1	"	"	$k' - c' = 2p,$
but	"	"	P_2	"	"	$k' - c' = 0.$

Use these equations to eliminate b_2 and c', and put $c a' = k a' - p$.

The result may be written thus:

$$p\,h\frac{\delta\,T}{\delta\,x} = \frac{k a' - p}{l}\sum_{A}^{B}(P_2 - P_1)\,b_1 - \frac{k a'}{l}\sum_{A}^{B}[P_2\,(b_1 - p) - P_1\,(b_1 + p)] - \sum_{A}^{C'}[P_2\,k' - P_1\,(k' - 2p)] + \sum_{A}^{C'}(P_2 - P_1)\,k' + \sum_{C'}^{K'}(P_2 - P_1)\,k',$$

in which the last two summations take the place of the last one in the previous equation by help of the identity, $A\,K' = A\,C' + C'\,K'$. Cancel and divide by p.

$$\therefore\ h\,\frac{\delta\,T}{\delta\,x} = \frac{k a'}{l}\sum_{A}^{B}(P_2 + P_1) - \sum_{A}^{B}(P_2 - P_1)\,\frac{b_1}{l} - 2\sum_{A}^{C'}P_1 + \sum_{C'}^{K'}(P_2 - P_1)\,\frac{k'}{p}.$$

But we have identically

$$\sum_{A}^{C'}(P_2 - P_1)\,\frac{(b_1 - a_1)}{l} - \sum_{A}^{C'}(P_2 + P_1) = -2\sum_{A}^{C'}P_1,$$

by help of which our equation is reduced to the form

$$h\,\frac{\delta\,T}{\delta\,x} = \frac{k a'}{l}\sum_{A}^{B}(P_2 + P_1) - \sum_{A}^{C'}(P_2 + P_1) + \sum_{C'}^{K'}(P_2 - P_1)\,\frac{k'}{p} - \sum_{A}^{C'}(P_2 - P_1)\,\frac{a_1}{l} - \sum_{C'}^{B}(P_2 - P_1)\,\frac{b_1}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

This is a general equation for a member of the loaded chord, and applies to a double intersection truss of any pattern. Let us first apply it to the truss shown in Fig. 7. In this case we may omit all primes, for $k = k'$, etc., and C and C' are on the same vertical, etc.,

Since $P_2 = 0$ between C' and K', we also have in this case,

$$\sum_{C}^{K'} (P_2 - P_1)\,\frac{k'}{p} = -\sum_{C}^{K} P_1$$

The result after reduction may be written in the form—

$$h\,\frac{\delta\,T}{\delta\,x} = \frac{k_a}{l}\sum_{A}^{B}(P_2 + P_1) - \sum_{A}^{K}(P_2 + P_1) - \sum_{A}^{C}(P_2 - P_1)\,\frac{a_1}{l}$$

$$- \sum_{C}^{B}(P_2 - P_1)\,\frac{b_1}{l},$$

or more briefly—

$$h\,\frac{\delta\,T}{\delta\,x} = k_a\left[\frac{L}{l} - \frac{K_a + C_1}{k_a}\right] \ldots\ldots\ldots\ldots (13_1)$$

in which equation as before—

L = the total load on the span $A\,B$.
K_a = " " " segment $K\,A$.

Also,

$$C_1 = \sum_{A}^{C}(P_2 - P_1)\,\frac{a_1}{l} + \sum_{C}^{B}(P_2 - P_1)\,\frac{b_1}{l} = -\frac{2}{l}\,(A_1 - B_1),$$

in which A_1 and B_1, have the values ascribed to them in Article 10, which are in general numerically small and separately vanish when the load is uniformly distributed. We may then regard C_1 as a small correction to be applied to K_a, and if it be disregarded for the moment, equation (13_1) would evidently give a train position identical with that before found to produce a maximum bending moment at K in a simple girder $A\,B$ in case $A\,K = A\,K' = k_a$. Now, in order to discover under what circumstances the last member of (13_1) may change sign from positive to negative, and so make T a maximum we must learn under what circumstances the last fraction in (13_1) receives a positive increment.

In case the joint K is in the right half of the span, let any wheel as W_u move from right to left of K, then is the quantity K_a increased by the wheel weight W_u, and the quantity C_1 is at the same time decreased by the amount $2\,W_u\,\frac{b_1}{l}$ (in which $b_1 = B\,K$), by reason of the passage of W_u from P_2 to P_1 in passing K. Hence the last numerator of (13_1) is increased on the whole by the amount—

$$W_u\,\frac{(l - 2\,b_1)}{l}.$$

This quantity is positive in the right half of the span, for we then have $l > 2\,b_1$, hence T may reach a maximum in the right half of the span just as some wheel W_u passes K, as it would not were K in the left half of the span.

In case the joint C is in the left half of the span, let any wheel as W_u pass from right to left of C, then will the left P_1 which is included in B_1, be decreased by W_u, and the right P_2 in A_1 will be increased by W_u. Hence the last numerator in equation (13_1) is increased on the whole by the quantity $W_u \frac{b_1 + a_1}{l}$ in which a_1 is negative, but $b_1 + a_1 > 0$, since C was taken in the left half of the span.

It is then evident that T may reach a maximum value in the left half of the span just as some wheel passes C, as it would not in the right half of the span.

Again, without regard to the right or left half of the span let any wheel as W_u pass some joint of component II at the right of K. Then W_u passes over from the P_1 at the right of this joint to the P_2 at its left, and at the same instant its b_1 is increased by $2p$. Hence such a movement suddenly increases C_1 by the quantity $2\,W_u \frac{p}{l}$, and this regardless of which joint of II the passage occurs at, provided only it is at the right of K. From this it is evident that T may reach a maximum just as a wheel passes any joint of II at the right of K, as it does not when it passes any joint of I at the right of K, for this latter change tends only to decrease B_1.

Once more, without regard to the position of the joint C, let some wheel as W_u, move from right to left of any joint of I which is situated at the left of C, then W_u passes over from P_2 at that joint to P_1. Hence such a movement suddenly increases the last numerator by the numerical amount $2\,W_u \frac{a_1}{l}$, an amount which is larger the nearer the joint is to C. From this we see that T may also reach a maximum value when a wheel passes any joint of I at the left of C, but is more likely to do so, other things being equal, the nearer the joint is to C.

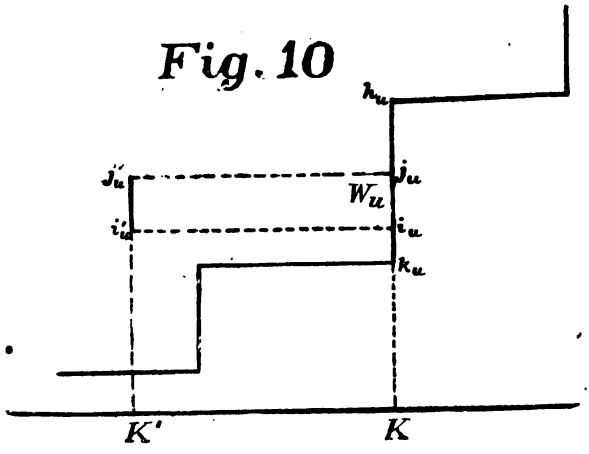
Since the quantity $A_1 - B_1$ appears in equations (11) and (13_1) with opposite signs, the conclusion drawn from (11) to the effect that the train progresses with respect to the panel where the shear is a maximum, must be drawn from equation (13_1) in the reverse way as to the chord stress T, to the effect that the train retrogrades on the whole with respect to the panel $C\,K$ in which the chord stress is a maximum as that panel is assumed further toward the left.

In a general way it may be said that the greater concentration of load is forward, *i. e.*, to the left of the panel under consideration in the right half of the span and behind it, *i. e.*, to the right of it, in the left half, or in other words, it is nearer the middle of the span than the panel is, at the

instant when the greatest stress occurs in the chord member of the panel.

We have then the following method of procedure for finding the train position which produces a maximum stress in $C\,K$.

1°. In case K is in the right half of the span, find by help of the weight line, as previously explained, the wheel W_u which, standing at K, would produce the greatest bending moment at K in a simple girder $A\,B$. The train position thus found is approximately that which produces a maximum stress in $C\,K$. Take W_u as standing just at the right of K and compute the value of C_1 for this train position. It will be shown a little later how this computation may be most expeditiously made by help of a graphical construction. The value thus obtained for C_1 is a positive or negative correction of K_a, and accordingly is to be laid off on the diagram of the weight line upward or downward from the lower extremity k_u of W_u. Let $k_u\,i_u$ be this correction in Fig. 10.



Now let W_u pass K from right to left, then will the last numerator of (13_1) be increased by the quantity

$$W_u \frac{l-2\,b_1}{l}$$

due to the passage of W_u across the point K. Let $i_u\,j_u$ represent this quantity in Fig. 10. Then may $i_u\,j_u$ be regarded as a fictitious weight such that, when it is transferred to the vertical at K' and used in the same manner as we have previously used actual weights for discovering maxima, it will enable us to discover the train position for a maximum value of T. In case K and K' are in the same vertical no transference is needed. Let the transferred position be $i_u'\,j_u'$, then it constitutes the fictitious weight in its true position on the diagram of the weight line, and when it is intersected between its extremities i_u' and j_u' by the slope

line $L \div l$ such intersection shows the train position to be one producing a maximum chord stress T in $C K$, but if not, not. This statement is evident since such intersection shows that (13_1) changes sign as W_u passes K. In case the slope line does not intersect the fictitious weight in the assumed position of the train, the train must be moved to the right or left until such intersection occurs just as some wheel passes some joint of II at the right of K, or some joint of I at the left of C.

It is more probable that the train must in this case be moved toward the left than toward the right, for reasons already given. On moving the train, whenever a wheel arrives at either of the joints just mentioned as possibly making T a maximum, a recomputation of (13_1) must be made. This will, in general, give the fictitious weight $i_u' j_u'$ a magnitude and position different from that before obtained.

Whenever these are such that it is cut between its extremities by the slope line $L \div l$, then the train has arrived at a position where it makes T a maximum. If there is more than one such position, the computed numerical values of T may be directly compared to discover which is greatest, as their general comparison algebraically appears too complicated to be of practical use.

2°. In case C is taken in the left half of the span, the method of procedure is the same as that just developed, except that it may be well to take as the initial position that which produces the maximum bending moment at C in a simple girder of span $A B$.

As shown before, the numerical value of $i_u' j_u'$ due to the passage of the weight W_u from right to left of C is

$$W_u \frac{a_1+b_1}{l}$$

If this fictitious weight is not intersected by the slope line $L \div l$, and so a maximum value of T does not occur in $C K$ as W_u passes C, then the train must be moved to the right or left and the proper train position found as before, but the train will more probably require to be moved to the right than to the left. But any uncertainty in this particular is mostly obviated in actual computation by finding the required train position for the panels successively from end to end of the truss.

For if the train position be found which produces a maximum chord stress in the left panel nearest A, say in $C_1 K_1$, then, as previously stated, the train position which gives a maximum in the next panel $C_2 K_2$ at the right of $C_1 K_1$ is such that the train may stand in general a little further to the left with respect to $C_2 K_2$ than it did with respect to $C_1 K_1$ when it produced a maximum in $C_1 K_1$, though this is of course not so far to the left with respect to $A B$. Hence, if W_u at C_1 makes T a maximum in $C_1 K_1$, and W_u at C_2 does not make T a maximum in $C_2 K_2$ then W_u must in general be moved toward the left to reach the train position which makes T a maximum in $C_2 K_2$, and so on for the other panels of this half of the truss.

The graphical computation of C_1 may be conveniently arranged, as shown in Fig. 11. Lay off on a piece of cross-section paper the span $A\,B$ and the panels $C\,K$, etc., of the truss to the same scale as the diagram of the wheel distances.

Then for any position of the truss at which it is desired to compute C_1 read off the value of $P_2 - P_1$ at each joint of I by the graphical method already given.

Read on the same scale or transfer directly, with compasses, the weight

$$P_2 - P_1 = A\,y_a$$

taken as positive upwards. Draw the thread or line $B\,y_a$; then is

$$(P_2 - P_1)\,\frac{b_1}{l} = K\,y_k$$

which can be read off to scale. In similar manner find at H

$$(P_2 - P_1)\frac{a_1}{l} = H\,y_h$$

which is to be taken downwards and negative when $P_2 - P_1 > 0$, because a_1 is essentially negative, and *vice versa*.

In this way values of all the quantities in the two summations which form C_1 in equation (13_1) may be found. After C_1 has been evaluated it can be used as proposed in the method already explained.

It is evident that in obtaining equation (13), b_1, k' and k_a' might have been eliminated instead of $b_2\,c'$ and c_a'. We shou'd thus have been led by a similar series of transformations to the following general equation equivalent to (13):

$$h\,\frac{\delta\,T}{\delta\,x} = \frac{c_a'}{l}\sum_{A}^{B}(P_2 + P_1) - \sum_{A}^{C'}(P_2 + P_1) + \sum_{C'}^{K'}(P_2 - P_1)\,\frac{k'}{p}$$

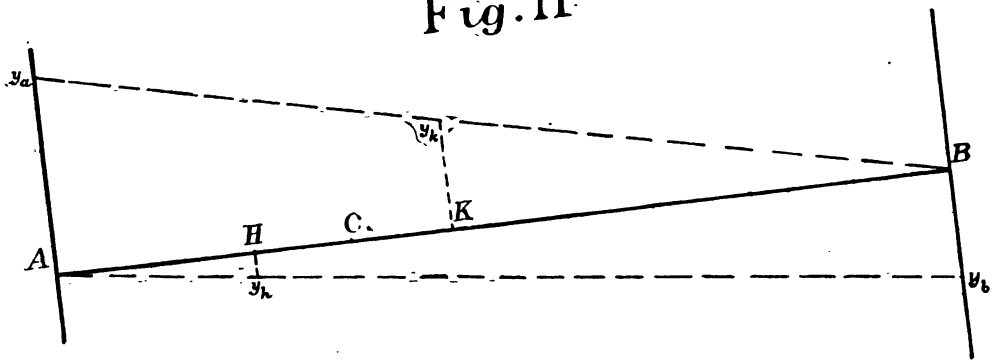
$$- \sum_{A}^{C'}(P_2 - P_1)\,\frac{a_2}{l} - \sum_{C'}^{B}(P_2 - P_1)\,\frac{b_2}{l} \quad \ldots\ldots\ldots (14)$$

which may be more convenient than equation (13) for graphical construction in some cases.

Equation (14) may be adapted to the truss in Fig. 7 by noticing that $\frac{k'}{p} = 1$ between C' and K', and making use also of the identity

$$\sum_{C'}^{K'}(P_2 - P_1) = \sum_{C}^{K}(P_2 - P_1)\,\frac{b_2 - a_2}{l}$$

Fig. 11



we obtain

$$h\frac{\delta T}{\delta x} = \frac{c_a}{l}\sum_{A}^{B}(P_2 + P_1) - \sum_{A}^{C}(P_2 + P_1)$$
$$- \sum_{A}^{K}(P_2 - P_2)\frac{a_2}{l} - \sum_{K}^{B}(P_2 - P_1)\frac{b_2}{l},$$

or

$$h\frac{\delta T}{\delta x} = c_a\left[\frac{L}{l} - \frac{C_a + K_2}{c_a}\right] \ldots\ldots\ldots\ldots\ldots\ldots (14_1)$$

which equation would serve to discover the set of conditions already found by means of (13_1).

14. To discover the train position which produces the greatest stress in any member of the unloaded chord of a double intersection truss, as in the bar $C'K'$ of Fig. 7:

Take the moment about K of all the forces acting on component I, and that about C of all the forces acting on II and we have

$$h\,T'_1 = V_1\,k_a - \sum_{A}^{C} R_1\,k$$

$$h\,T'_2 = V_2\,c_a - \sum_{A}^{H} R_2\,c$$

Now $T' = T'_1 + T'_2$ and $c_a = k_a - p$, in which equations primes are used to refer to the unloaded chord; also k is the distance of any R_1 from K and c that of any R_2 from C.

Adding the equations of moments and substituting the values of the reactions and pin loads as was done in the previous case we have

$$p\,h\,T' = \frac{k_a}{l}\sum_{A}^{B}(P_2\,c_2 - P_1\,c_1)\,b_1 - \frac{c_a}{l}\sum_{A}^{B}(P_2\,k_2 - P_1\,k_1)\,b_2$$
$$- \sum_{A}^{C}(P_2\,c_2 - P_1\,c_1)\,k + \sum_{A}^{H}(P_2\,k_2 - P_1\,k_1)\,c,$$

Move the train a small distance δx toward the left,

Then for each	P_1,	$k - c$	$= 0$
but " "	P_2,	$k - c$	$= 2$
while " "	P_1,	$b_2 - b_1$	$= p$
but " "	P_2,	$b_1 - b_2$	$= p$

By eliminating b_1 and $c_a = k_a - p$ we find after reductions like those used in obtaining equation (13)

$$h\frac{\delta T'}{\delta x} = \frac{k_a}{l}\sum_{A}^{B}(P_1+P_2) - \sum_{A}^{H}(P_1+P_2) + \sum_{H}^{C}(P_1-P_2)\frac{k}{p}$$
$$-\sum_{A}^{H}(P_1-P_2)\frac{a_2}{l} - \sum_{H}^{B}(P_1-P_2)\frac{b_2}{l} \quad \ldots\ldots\ldots(15)$$

If, however, b_2 and k_a be eliminated the result obtained is,

$$h\frac{\delta T'}{\delta x} = \frac{c_a}{l}\sum_{A}^{B}(P_1+P_2) - \sum_{A}^{H}(P_1+P_2) + \sum_{H}^{C}(P_1-P_2\frac{k}{p}$$
$$-\sum_{A}^{H}(P_1-P_2)\frac{a_1}{l} - \sum_{H}^{B}(P_1-P_2)\frac{b_1}{l} \quad \ldots\ldots\ldots(16)$$

Equations (15) and (16) are general and apply to any double intersection truss.

Equation (15) may be adapted to the truss in Fig. 7, by noticing that in the third term $\frac{k}{p}=2$ and $P_1=0$ between H and C, also $P_2=0$ between C and K. Hence we may write the identities

$$2\sum_{H}^{C}(P_1-P_2) = \sum_{H}^{C}(P_1-P_2)\frac{b_2-a_2}{l} - \sum_{H}^{C}(P_1+P_2),$$
$$0 = \sum_{C}^{K}(P_1-P_2)\frac{(b_2-a_2)}{l} - \sum_{C}^{K}(P_1+P_2).$$

Hence equation (15) becomes

$$h\frac{\delta T'}{\delta x} = \frac{k_a}{l}\sum_{A}^{B}(P_1+P_2) - \sum_{A}^{K}(P_1+P_2)$$
$$-\sum_{A}^{K}(P_1-P_2)\frac{a_2}{l} - \sum_{K}^{B}(P_1-P_2)\frac{b_2}{l},$$

or,

$$h\frac{\delta T'}{\delta x} = k_a\left[\frac{L}{l} - \frac{K_a-K_2}{k_a}\right] \ldots\ldots\ldots\ldots(15_1)$$

in which the notation is obvious.

Equation (16) may be similarly transformed to

$$h\frac{\delta T'}{\delta x} = \frac{c_a}{l}\sum_{A}^{B}(P_1+P_2) - \sum_{A}^{C}(P_1+P_2)$$
$$-\sum_{A}^{C}(P_1-P_2)\frac{a_1}{l} - \sum_{C}^{B}(P_1-P_2)\frac{b_1}{l},$$

or,

$$h\frac{\delta T'}{\delta x} = c_a\left[\frac{L}{l} - \frac{C_a-C_1}{c_a}\right] \ldots\ldots\ldots\ldots(16_1)$$

These equations are to be interpreted in a manner similar to (13_1) and determine position of the train which produces a maximum in $C'K'$ of Fig. 7, in the manner already discussed in connection with that equation.

15. To discover the train position which produces a maximum stress in any chord member of the double intersection truss with intermediate supports, Fig. 9:

The expression for the total tension T in any member CK, Fig. 9, differs from that already obtained for the stress in the same member CK, Fig. 7, only in terms depending upon the magnitude and position of the wheel-weights between C and K. The truth of this statement may be made to appear as follows: the total reaction V at the pier A depends solely upon the magnitude and position of the wheel-weights between A and B, and not upon the arrangement of the members of the truss. Now take the moments about Q' situated in the vertical at J of all the external forces applied to the truss at the left of Q'. So far as V and all the wheel-weights at the left of C are concerned we are at liberty in computing their moments about Q' to take either the wheel-weights themselves or the pin loads; but in considering the wheels between C and K it is necessary to take into account how much of this panel load is supported at C.

It is assumed that this has been done correctly and has been included in the value of $h\,T$ obtained in the case of Fig. 7, and that we should find the same value of $h\,T$ in Fig. 7 if it were computed by taking moments about any point Q' between C' and K' of all the forces applied at the left of Q', provided we should also correctly take account of the stresses in the web members of the panel. It was the difficulty of attending properly to this last matter that has led to the method of separately computing T' and T'', and then taking their sum to obtain T.

The value of $h\,T$ is, then, equal to the algebraic sum of the moments of all the forces at the left of Q' minus the moment about Q' of the stresses in the members of the web cut by the vertical at Q'. As already stated, the moment of the applied forces differs in the two cases only in so far as the panel load CK is concerned, and the moment of the stresses in the web members differs not at all in the two Figs. 7 and 9. This last assertion may be seen to be true from the following considerations. Whatever changes are produced in the stresses of the web members meeting at J, when Q is supported from J, it is evident from their symmetrical disposition about the vertical at J, that the part of the load applied at J, which is transmitted to the pier at A causes as great a tension in $C'J$ as compression in CJ. Hence their moment of resistance about Q' vanishes. Now, that part of the value of $h\,T$ in Fig. 7, which depends upon the panel load of CK is the part of the pin load at C due to this panel, which is $\frac{P_1\,k}{p}$, multiplied by its lever

arm from Q, which $Q\ C = \frac{1}{2}\ p$. Hence the part of $h\ T$, in Fig. 7, due to the panel $C\ K$, is $\frac{1}{2}\ P_1\ k$.

Again, that part of $h\ T$, in Fig. 9, which depends upon the panel load of $C\ K$ is that part of the pin load at C due to this panel, which is $\frac{P_1'\ q}{\frac{1}{2}\ p}$, multiplied by its arm, $\frac{1}{2}\ p$, or, in all, $P_1'\ q$. The whole effect will then be to replace the last term in the value $p\ h\ T'$ in Art. 13, viz.:

$$\sum_{A}^{K'} (P_2\ k_2 - P_1\ k_1)\ k'$$

by the following expression:

$$\sum_{A}^{C} (P_2\ k_2 - P_1\ k_1)\ k' - \sum_{C}^{Q} P_1'\ q\ k'$$

so that we must then, in equation (13), replace

$$K_a = \sum_{A}^{K} (P_2 + P_1) = \sum_{A}^{K} W$$

by the following:

$$Q_a = \sum_{A}^{Q} W$$

The introduction of Q_a instead of of K_a will simply have the effect of making the point Q play the rôle ascribed in the discussion of equation (13_1) to the point K.

It will be seen from these expressions that the introduction of intermediate pin points has the effect to increase slightly the tension of the loaded chord for a given train position; but at the same time it materially decreases the bending moments of the stringers, since they are decreased in length to $\frac{1}{2}\ p$. The stresses in the web members are also slightly increased. There is also an increase of material due to the introduction of the pin points at Q and J and the pieces of the type $Q\ J$.

The stress T' in any member $C'\ K'$ of the unloaded chord is unchanged by the introduction of intermediate joints in the loaded chord.

16. To discover the position of the train which produces the greatest stress in any given bar of the loaded chord of a double intersection truss with vertical web members, as in the Whipple truss.

Since the stress is identical in the members of the upper and lower chords respectively, which lie between any pair of consecutive diagonals, it will in general be possible to determine the train position for a maximum stress in any bar of either chord from either of the equations (13), (14), (15) or (16).

In trying to determine the train position for a Whipple Truss, which possesses a system of counters as well as principal web members, it might not at first appear which system should be considered as acting.

But in case the stress in a bar of the loaded chord is at a maximum it is sufficiently evident that the principal web members between that bar and the nearest pier are in action, a statement which will be seen to be in accordance with that made in connection with Fig. 7, where it was shown that to produce a maximum stress in a bar the greatest concentration of load lies between it and the center of the span, which is true in general for any truss with end supports.

1°. In order to find the train position for a maximum stress in $C\,K$ when C is in the left half of the truss, let the inclinations of the web members shown in Fig. 7 be changed until $H\,C'$ and $C\,K'$ become vertical, etc. The reader will readily supply the appropriate figure. Now apply the general equation (13) to this case, in which we first replace the limit C' by K and $k_a' = c_a$ and notice that the third term vanishes since $k' = 0$ between C' and K', *i. e.*, between H and C; also add the identity

$$0 = \sum_{H}^{C} (P_2 - P_1)\,\frac{b_1 - a_1}{l} - \sum_{H}^{C} (P_2 + P_1),$$

which evidently is true because $P_1 = 0$ between H and C.

After reductions such as we have made before we have

$$h\,\frac{\delta\,T}{\delta\,x} = \frac{c_a}{l}\sum_{A}^{B}(P_2 + P_1) - \sum_{A}^{C}(P_2 + P_1) - \sum_{A}^{C}(P_2 - P_1)\,\frac{a_1}{l} - \sum_{C}^{B}(P_2 - P_1)\,\frac{b_1}{l},$$

or,

$$h\,\frac{\delta\,T}{\delta\,x} = c_a\left[\frac{L}{l} - \frac{C_a + C_1}{c_a}\right] \qquad (13_2)$$

We find that essentially the same result can be obtained from equation (15) which refers to $C'\,K'$ or to its corresponding bar in the lower chord, viz.: that one extending from K towards the right. The equation we then find puts k_a for c_a, K for C, a_2 and b_2 for a_1 and b_1, and exchanges P_1 and P_2 with each other, as should be done in applying (13_2), an expression referring to $C\,K$, to the next panel at the right of $C\,K$.

Now, let some wheel W_u pass the point C from right to left, then is C_a increased by the amount W_u and C_1 by the amount $W_u\,\frac{a_1 + b_1}{l}$, both which quantities are positive though a_1 is negative. Hence T may reach a maximum as W_u passes C. In the graphical construction, similar to Fig. 10, let $k_u\,h_u$ be the vertical part of the weight line representing W_u, prolong $k_u\,h_u$ upward to a height

$$h_u\,n_u = W_u\,\frac{a_1 + b_1}{l}.$$

Then if the slope line $L \div l$ cuts $k_u\,n_u$, the wheel W_u at C produces a

maximum stress in the bar $C\,K$, for $k_u\,n_u$ is the amount by which $C_a + C_1$ changes as W_u passes C.

Again, whenever any wheel W_u passes from right to left of any joint of the same component system as C at the right of C, we find that C_1 is increased by the amount

$$2\,W_u\,\frac{p}{l}$$

by reason of the transference of W_u from P_1 to P_2 and the simultaneous sudden increase in the co-ordinate b_1 by the amount $2\,p$. Hence a maximum value of T' may occur when any wheel passes any joint at the right of C and belonging to the same system as C, as it does not when any wheel passes any other joint at the right of C, including K.

Again, it appears in like manner that when W_u passes any joint of system I at the left of C, the increment of C_1 is the positive quantity

$$-2\,W_u\,\frac{a_1}{l}.$$

Hence the maximum stress in $C\,K$ cannot occur as any wheel passes any joint of the same system as C at the left of C, as it may when it passes any other joint at the left of C.

The nearer these last joints lie to $C\,K$ the larger are these increments, and consequently the more likely are they to produce maximum stresses in $C\,K$.

2°. In order to find the train position for a maximum stress in $C\,K$ when K lies in the right half of the span, let the inclinations of the web members shown in Fig. 7 be altered so that $C\,H'$ and $K\,C'$ are vertical, etc.

Now, apply the general equation (14) to this case, and after we have replaced the limit C' by K and put $c_a' = k_a$, and noticed that the third term vanishes as before, we obtain at once

$$h\,\frac{\delta\,T}{\delta\,x} = \frac{k_a}{l}\sum_{A}^{B}(P_2 + P_1) - \sum_{A}^{K}(P_2 + P_1) - \sum_{A}^{K}(P_2 - P_1)\,\frac{a_2}{l}$$

$$-\sum_{K}^{B}(P_2 - P_1)\,\frac{b_2}{l}$$

$$\text{or,}\qquad h\,\frac{\delta\,T}{\delta\,x} = k_a\left[\frac{L}{l} - \frac{K_a + K_2}{k_a}\right] \qquad \ldots\ldots\ldots\ldots (14_2)$$

We obtain a result which is essentially the same as this in applying equation (16) to find the position for a maximum stress in $C'\,K'$ equal to that in $H\,C$, with such changes of notation as fit it to $H\,C$ instead of $C\,K$.

It appears from equation (14_2) by making use of considerations precisely like those employed in connection with (13_2) that a maximum may occur in $C\ K$ in the right half of $A\ B$, when a wheel passes K.

It also may occur when any wheel W_u passes any joint of the same system as K at the left of K, the increment of K_2 then being

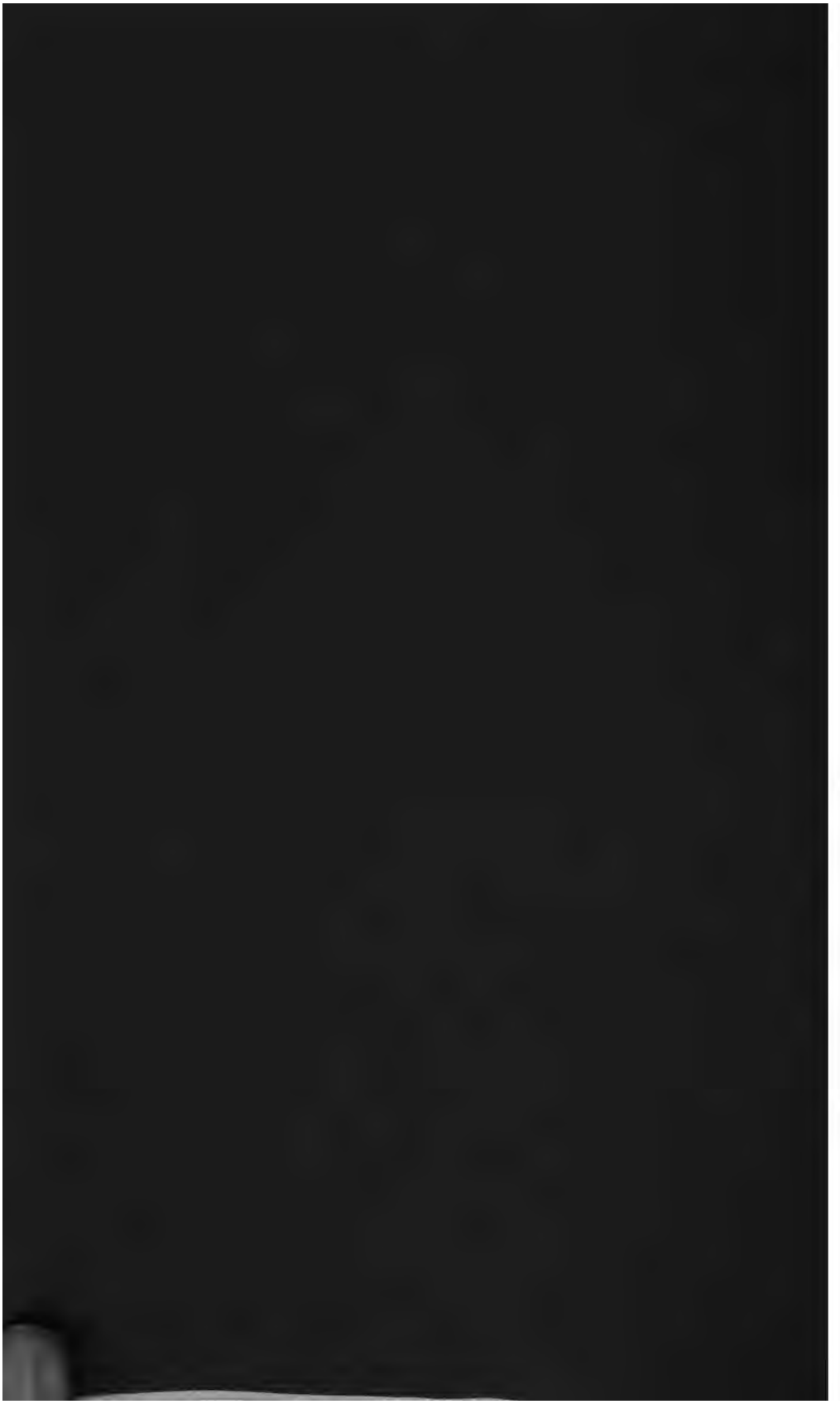
$$2\ W_u \frac{p}{l},$$

and finally it may occur when any wheel W_u passes any joint not of the same system as K, situated at the right of K, the increment then being

$$2\ W_u \frac{b_2}{l}.$$

These statements as to the positions which may cause maximum stresses in $C\ K$ in the right half of the span could have been made by symmetry from those already made respecting the left half of the span.

The various practical devices to be employed in the graphical application of equations (13_2) and (14_2) have been already sufficiently explained in connection with the discussion of equation (13_1).



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